On The Inequalities Similar to the Hilbert's Inequality

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Abstract. By introducing the function

\[
\frac{x^{\alpha} y^{\alpha}}{x^\alpha + y^\alpha + \min\{x^\alpha, y^\alpha\}} \ln x^\alpha - \ln y^\alpha \geq \frac{1}{\alpha},
\]

we study new inequalities similar to Hilbert's type inequality. We also consider its equivalent form as well.

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1. Introduction

If \( f(x), \ g(x) \geq 0 \), such that \( 0 < \int_0^\infty f^2(x)dx < \infty \) and \( 0 < \int_0^\infty g^2(x)dx < \infty \) then

\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dydx < \pi \left( \int_0^\infty f^2(x)dx \right)^{\frac{1}{2}} \left( \int_0^\infty g^2(x)dx \right)^{\frac{1}{2}}, \tag{1.1}
\]

where the constant factor \( \pi \) is the best possible (see [4],[5]). Inequality (1.1) had been extended by Hardy-Riesz as :

If \( p > 1 \), \( \frac{1}{p} + \frac{1}{q} = 1 \), \( f(x), \ g(x) \geq 0 \), such that \( 0 < \int_0^\infty f^p(x)dx < \infty \) and \( 0 < \int_0^\infty g^q(x)dx < \infty \), then we have the following Hardy-Hilbert's integral inequality:

\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dydx < \frac{\pi}{\sin^2\left(\frac{\pi}{2}\right)} \left( \int_0^\infty f^p(x)dx \right)^{\frac{1}{p}} \left( \int_0^\infty g^q(x)dx \right)^{\frac{1}{q}}, \tag{1.2}
\]

where the constant factor \( \frac{\pi}{\sin^2\left(\frac{\pi}{2}\right)} \) is the best possible constant (see [2]). This inequality play an important role in mathematical analysis, which is named of Hardy-Hilbert's inequality and its applications(see [11]), it has been studied and generalized in many directions by a number of mathematicians(see [1-3],[7,8],[12-16]).

Under the some condition of (1.2), we have the Hardy-Hilbert's type inequality (cf. Hardy et al.[4])

\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\min\{x,y\}} dydx < 4 \left( \int_0^\infty f^2(x)dx \right)^{\frac{1}{2}} \left( \int_0^\infty g^2(x)dx \right)^{\frac{1}{2}}, \tag{1.3}
\]

\[
\int_0^\infty \int_0^\infty \frac{\ln x - \ln y}{x-y} f(x)g(y) dydx < \pi^2 \left( \int_0^\infty f^2(x)dx \right)^{\frac{1}{2}} \left( \int_0^\infty g^2(x)dx \right)^{\frac{1}{2}}. \tag{1.4}
\]
where the constant factors $4$ and $\pi$ are both the best possible.

Li, Wu and He [9] obtained the following inequality:

**Theorem 1.1.** If $f, g$ are real function such that $0 < \int_0^\infty f^2(x)dx < \infty$ and $0 < \int_0^\infty g^2(x)dx < \infty$. Then we have

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y+\min\{x,y\}} dxdy \leq C\left(\int_0^\infty f^2(x)dx\int_0^\infty g^2(x)dx\right)^{\frac{1}{2}}, \quad (1.5)$$

where the constant factor $C = 1.7408...$ is the best possible.

Recently Y.Li, Q. You and B. He [10] obtained the following inequality:

**Theorem 1.2.** If $f, g$ are real function such that $0 < \int_0^\infty f^2(x)dx < \infty$ and $0 < \int_0^\infty g^2(x)dx < \infty$. Then we have

$$\int_0^\infty \int_0^\infty \frac{\ln(x) - \ln(y)}{x+y+\min\{x,y\}} f(x)g(y)dxdy \leq A\left(\int_0^\infty f^2(x)dx\int_0^\infty g^2(x)dx\right)^{\frac{1}{2}}, \quad (1.6)$$

where $C = 7.3277...$.

B. He, Q. You and Y.Li [6] gave a generalization and improvement of Hilbert’s inequality as following:

**Theorem 1.3.** If $f, g$ are real function such that $0 < \int_0^\infty f^2(x)dx < \infty$ and $0 < \int_0^\infty g^2(x)dx < \infty$. Then, we have

$$\int_0^\infty \int_0^\infty \frac{\ln(x) - \ln(y)}{x+y+\min\{x,y\}} f(x)g(y)dxdy \leq A\left(\int_0^\infty f^2(x)dx\int_0^\infty g^2(x)dx\right)^{\frac{1}{2}}, \quad (1.7)$$

where the constant factor $A = 6.88947...$ is the best possible.

The object of this paper is that to give a generalization and improvement of Hilbert’s inequality by (1.4), (1.5), (1.6) end (1.7) as following.

$$\int_0^\infty \int_0^\infty \frac{\ln(x) - \ln(y)}{x+y+\min\{x,y\}} f(x)g(y)dxdy \leq A\left(\int_0^\infty f^2(x)dx\int_0^\infty g^2(x)dx\right)^{\frac{1}{2}}, \quad (1.8)$$

where $A = \frac{6.88947...}{\alpha}$ and $\alpha \geq 1$.

### 2. Main Results

**Theorem 2.1.** If $f, g$ are real function such that $0 < \int_0^\infty f^2(x)dx < \infty$ and $0 < \int_0^\infty g^2(x)dx < \infty$. Then we have

$$\int_0^\infty \int_0^\infty \frac{\ln(x) - \ln(y)}{x+y+\min\{x,y\}} f(x)g(y)dxdy \leq A\left(\int_0^\infty f^2(x)dx\int_0^\infty g^2(x)dx\right)^{\frac{1}{2}}, \quad (2.1)$$

where $A = \frac{6.88947...}{\alpha}$ is the best possible dependent on $\alpha \geq 1$.  


Proof. By Hölder’s inequality we have the following inequality.

\[
\int_{0}^{\infty} \int_{0}^{\infty} x^\alpha y^{\alpha-1} \left| \ln x^\alpha - \ln y^\alpha \right| \frac{f(x)g(y)dxdy}{x^\alpha + y^\alpha + \min\{x^\alpha, y^\alpha\}} \\
= \int_{0}^{\infty} \int_{0}^{\infty} \left[ y^{\alpha-1} \left| \ln x^\alpha - \ln y^\alpha \right| \right] \frac{1}{x^\alpha + y^\alpha + \min\{x^\alpha, y^\alpha\}} \left( \frac{1}{\gamma} \right)^\frac{\gamma}{2} f(x) \\
\times \left[ \frac{1}{x^\alpha + y^\alpha + \min\{x^\alpha, y^\alpha\}} \left( \frac{1}{\gamma} \right)^\frac{\gamma}{2} g(y) \right] dxdy \\
\leq \int_{0}^{\infty} \int_{0}^{\infty} \left[ y^{\alpha-1} \left| \ln x^\alpha - \ln y^\alpha \right| \right] \frac{1}{x^\alpha + y^\alpha + \min\{x^\alpha, y^\alpha\}} \left( \frac{1}{\gamma} \right)^\frac{\gamma}{2} dy \left( \frac{1}{\gamma} \right)^\frac{\gamma}{2} dx \\
\times \left[ \frac{1}{x^\alpha + y^\alpha + \min\{x^\alpha, y^\alpha\}} \left( \frac{1}{\gamma} \right)^\frac{\gamma}{2} \right] g^2(y)dy \right]^\frac{1}{2}.
\]

Define the weight function \( \omega(u) \) as

\[
\omega(u) := \int_{0}^{\infty} \left| \ln u^\alpha - \ln v^\alpha \right| \frac{1}{u^\alpha + v^\alpha + \min\{u^\alpha, v^\alpha\}} \left( \frac{1}{\gamma} \right)^\frac{\gamma}{2} dv.
\]

For fixed \( u \), let \( v = ut \), we have the following equality

\[
\omega(u) = \int_{0}^{\infty} \frac{(ut)^{\alpha-1} |\ln u^\alpha - \ln (ut)^\alpha|}{(ut)^\alpha + u^\alpha + \min\{u^\alpha, v^\alpha\}} \left( \frac{1}{\gamma} \right)^\frac{\gamma}{2} u dt \\
= \int_{0}^{\infty} \frac{t^{\alpha-1} |\ln t^\alpha|}{1 + t^\alpha + \min\{1, t^\alpha\}} \left( \frac{1}{\gamma} \right)^\frac{\gamma}{2} dt \\
= \frac{1}{\alpha} \int_{0}^{\infty} \frac{|\ln z|}{1 + z + \min\{1, z\}} \left( \frac{1}{\gamma} \right)^\frac{\gamma}{2} dz, \quad z = t^\alpha \\
= \frac{1}{\alpha} \left[ \int_{0}^{\infty} \frac{|\ln z|}{1 + z + \min\{1, z\}} \left( \frac{1}{\gamma} \right)^\frac{\gamma}{2} dz + \int_{1}^{\infty} \frac{|\ln z|}{1 + z + \min\{1, z\}} \left( \frac{1}{\gamma} \right)^\frac{\gamma}{2} dz \right] \\
= \frac{1}{\alpha} \left[ -\frac{1}{\alpha} \int_{0}^{\infty} \frac{|\ln z|}{1 + 2z} \left( \frac{1}{\gamma} \right)^\frac{\gamma}{2} dz + \frac{1}{\gamma} \ln z \left( \frac{1}{\gamma} \right)^\frac{\gamma}{2} \frac{\gamma}{2} \right] \\
= -\frac{1}{\alpha} \frac{1}{\gamma} \frac{1}{\gamma} \frac{\gamma}{2} dz \\
= -\frac{1}{\alpha} \frac{1}{\gamma} \frac{\gamma}{2} zs, \quad s = z^\frac{1}{\alpha} \\
= \frac{6.88947}{\alpha}.
\]

Thus

\[
\int_{0}^{\infty} \int_{0}^{\infty} x^\alpha y^{\alpha-1} \left| \ln x^\alpha - \ln y^\alpha \right| \frac{f(x)g(y)dxdy}{x^\alpha + y^\alpha + \min\{x^\alpha, y^\alpha\}} \leq A \left( \int_{0}^{\infty} f^2(x)dx \int_{0}^{\infty} g^2(y)dy \right)^\frac{1}{2}.
\]
If (2.2) take the form of the equality, then there exist constants $\delta$ and $\beta$, such that they are not all zero ( Without loss of generality, suppose that $\delta \neq 0$ ) and

$$\delta \frac{y^{\alpha}}{x^\alpha + y^\alpha + \min \{x^\alpha, y^\alpha\}} \left( \frac{x}{y} \right)^{\beta} f^2(x) = \beta \frac{x^{\alpha}}{x^\alpha + y^\alpha + \min \{x^\alpha, y^\alpha\}} \left( \frac{x}{y} \right)^{\beta} g^2(y)$$

a.e. in $(0, \infty) \times (0, \infty)$. Then we have

$$\delta f^2(x) = \beta y g^2(y), \text{ a.e. in } (0, \infty) \times (0, \infty).$$

Hence we obtain

$$\delta f^2(x) = \beta y g^2(y) = \text{const} \quad \text{a.e. in } (0, \infty) \times (0, \infty).$$

Thus

$$\int_0^\infty f^2(x)dx = \frac{1}{\delta} \int_0^\infty dx = \frac{1}{\delta} \int_0^\infty dx,$$

which contradicts the facts that $0 < \int_0^\infty f^2(x)dx < \infty$. Hence (2.2) takes the form of strict inequality. So we have (2.1).

Assume that the constant factor $A = -\frac{1}{2} \ln s \frac{a}{a+2s} ds \in (2.1)$ is not the best possible, then exists a positive number $K$ kith $K < A$ and $a > 0$, we have

$$\int_{a}^{\infty} x^{\frac{\alpha}{2}} y^{\frac{\alpha}{2}} \left| \ln x^\alpha - \ln y^\alpha \right| f(x) g(y) dx dy < K \left( \int_{a}^{\infty} f^2(x) dx \int_{a}^{\infty} g^2(x) dx \right)^{\frac{1}{2}} \quad (2.3)$$

For $0 < \varepsilon < 1$, setting $b > 0 (b < a)$, $f_\varepsilon(x) = x^{-\varepsilon \frac{\alpha}{2}}$, for $x \in (b, \infty)$; $f_\varepsilon(x) = 0$, for $x \in (0, b)$. $g_\varepsilon(y) = y^{-\varepsilon \frac{\alpha}{2}}$, for $y \in (b, \infty)$; $g_\varepsilon(y) = 0$, for $y \in (0, b)$. Since,

$$K \left( \int_{a}^{\infty} f^2(x) dx \int_{a}^{\infty} g^2(x) dx \right)^{\frac{1}{2}} = K \int_{a}^{\infty} x^{-1} dx = K \frac{1}{\varepsilon a^\alpha},$$

setting $y = ux$, we find

$$\int_{a}^{\infty} x^{\frac{\alpha}{2}} y^{\frac{\alpha}{2}} \left| \ln x^\alpha - \ln y^\alpha \right| f_\varepsilon(x) g_\varepsilon(y) dx dy = \int_{a}^{\infty} x^{\frac{\alpha}{2}} y^{\frac{\alpha}{2}} \left| \ln x^\alpha - \ln y^\alpha \right| x^{-\varepsilon \frac{\alpha}{2}} y^{-\varepsilon \frac{\alpha}{2}} dx dy = \int_{a}^{\infty} u^{-\varepsilon \frac{\alpha}{2}} \left| \ln u^\alpha \right| y^{-\varepsilon \frac{\alpha}{2}} x^{-\varepsilon \frac{\alpha}{2}} dx du.$$

By (2.3) and for $b \to 0^+$, we have

$$\int_{a}^{\infty} u^{-\varepsilon \frac{\alpha}{2}} \left| \ln u^\alpha \right| x^{-\varepsilon \frac{\alpha}{2}} u^{-\varepsilon \frac{\alpha}{2}} dx du \leq K \frac{1}{a^\alpha}.$$
or
\[
\int_0^\infty \frac{u^{a-1} |\ln u^a|}{1 + u^a + \min \{1, u^a\}} u^{\frac{2}{\alpha}} du \leq K
\]
when \( \varepsilon \to 0^+ \), we have the following equality
\[
\int_0^\infty \frac{u^{a-1} |\ln u^a|}{1 + u^a + \min \{1, u^a\}} u^{\frac{2}{\alpha}} du = \int_0^\infty \frac{u^{\frac{2}{\alpha}} |\ln u^a|}{1 + u^a + \min \{1, u^a\}} du + 0(1) = A + 0(1).
\]

It follows the \( A \leq K \), which contradicts the hypothesis. Hence the constant factor \( A \) in (2.1) is the best possible depent on \( \alpha \geq 1 \).

**Theorem 2.2.** Suppose \( f \geq 0 \) and \( 0 < \int_0^\infty f^2(x)dx < \infty \). Then
\[
\int_0^\infty \int_0^\infty \left[ \frac{x^{a-1} y^{a-1}}{x^a + y^a + \min \{x^a, y^a\}} |\ln x^a - \ln y^a| f(x)dx \right]^2 dy < A^2 \int_0^\infty f^2(x)dx \quad (2.4)
\]
where the constant factor \( A^2 \) is the best possible depent on \( \alpha \geq 1 \). Inequality (2.4) is equivalent to (2.1).

**Proof.** Let
\[
g(y) = \int_0^\infty \frac{x^{a-1} y^{a-1}}{x^a + y^a + \min \{x^a, y^a\}} |\ln x^a - \ln y^a| f(x)dx,
\]
then by (2.2), we have the following inequality.
\[
0 < \int_0^\infty g^2(y)dy = \int_0^\infty \int_0^\infty \left[ \frac{x^{a-1} y^{a-1}}{x^a + y^a + \min \{x^a, y^a\}} |\ln x^a - \ln y^a| f(x)dx \right]^2 dy
\]
\[
= \int_0^\infty \int_0^\infty \frac{x^{a-1} y^{a-1}}{x^a + y^a + \min \{x^a, y^a\}} f(x)g(y)dydx
\]
\[
\leq \left( \int_0^\infty f^2(x)dx \right) \left( \int_0^\infty g^2(x)dx \right)^\frac{1}{2}.
\]
Hence we have
\[
\int_0^\infty \int_0^\infty \frac{x^{a-1} y^{a-1}}{x^a + y^a + \min \{x^a, y^a\}} |\ln x^a - \ln y^a| f(x)dx \leq A^2 \int_0^\infty f^2(x)dx < \infty. \quad (2.6)
\]
By (2.1), both (2.5) and (2.6) take the form of strict inequality, so we have (2.4). On the other hand, suppose that (2.4) is valid. By Hölder’s inequality, we find
By (2.4), we have (2.1). Thus (2.1) and (2.4) equivalent. If the constant $A^2$ in (2.4) not the best possible, by (2.7), we may get a contradiction that the constant factor in (2.1) is not the best possible. This completes the proof.

References