Computable points in co-c.e. polyhedra
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Abstract
Co-c.e. sets need not be computable, moreover they need not contain any computable point. Co-c.e. polyhedra also do not have to be computable. However, each co-c.e. polyhedron $P$ contains computable points, in fact we prove that computable points are dense in $P$.

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INTRODUCTION

A closed subset $S$ of $\mathbb{R}^n$ is co-c.e. (co-computably enumerable) if the complement of $S$ can be effectively covered by rational open balls. A closed subset $S$ of $\mathbb{R}^n$ is computable if $S$ can be effectively approximated by a finite set of rational points with arbitrary given precision on arbitrary given bounded region of $\mathbb{R}^n$. Each computable set is co-c.e., but a co-c.e. set need not be computable. Moreover, while computable points in a computable set are dense, there exists a nonempty co-c.e. set which does not contain any computable point [10].

However, there are certain conditions under which a co-c.e. set is computable and there are also certain conditions under which a (not necessarily computable) co-c.e. set contains computable points. In particular, if $S$ is a co-c.e. set in $\mathbb{R}^n$ which is homeomorphic to the unit closed ball $B^m$ in $\mathbb{R}^m$ for some $m \in \mathbb{N}$, then $S$ need not be computable, but $S$ contains computable points, moreover they are dense in $S$ [7].

Some other properties of a co-c.e. set $S$ can also assure that $S$ has a computable point or that $S$ is computable, for example if $S$ is a topological sphere [7], graph of a certain function [1], chainable or circularly chainable continuum [4], compact manifold with boundary [5] or 1-manifold with boundary [3].

On the other hand, in [6] is constructed a contractible co-c.e. set in $\mathbb{R}^2$ which does not contain any computable point.

In this paper we observe polyhedra. A polyhedron is a subset of Euclidean space obtained by gluing simplices along their faces in an appropriate way. Each simplex is an polyhedron, in particular each line segment is a polyhedron and by [7] there exists a co-c.e. line segment which is not computable. Hence co-c.e. polyhedra need not be computable. Using results from [7] and some geometric properties of polyhedra we prove that each co-c.e. polyhedron contains computable points and that they are dense in it.

PRELIMINARIES

Let $k \in \mathbb{N} \setminus \{0\}$. A function $F : \mathbb{N}^k \rightarrow \mathbb{Q}$ is called computable if there exist computable (recursive) functions $a, b, c : \mathbb{N}^k \rightarrow \mathbb{N}$ such that

$$F(x) = (-1)^{a(x)} \frac{a(x)}{b(x) + 1}$$

for each $x \in \mathbb{N}^k$ [9].

A number $x \in \mathbb{R}$ is said to be computable if there exists a computable function $g : \mathbb{N} \rightarrow \mathbb{Q}$ such that

$$|x - g(i)| < 2^{-i}$$
for each \( i \in \mathbb{N} \). A point \((x_1, \ldots, x_n)\) in \( \mathbb{R}^n\) is said to be computable if \( x_1, \ldots, x_n\) are computable numbers.

Let \( n \in \mathbb{N} \setminus \{0\} \). A sequence \((x_i)\) in \( \mathbb{Q}^n\) is said to be computable if the component sequences of \((x_i)\) are computable (as sequences in \( \mathbb{Q}\), i.e. as functions \( \mathbb{N} \to \mathbb{Q}\)).

Let \( S \) be a closed subset of \( \mathbb{R}^n\). We say that \( S \) is co-computably enumerable (co-c.e.) \([2]\) if \( S = \mathbb{R}^n \) or if there exists a computable sequence \((x_i)\) in \( \mathbb{Q}^n\) and a computable sequence \((r_i)\) in \( \mathbb{Q}\) such that \( r_i > 0 \) for each \( i \in \mathbb{N}\) and such that

\[
\mathbb{R}^n \setminus S = \bigcup_{i \in \mathbb{N}} B(x_i, r_i).
\]

Here, for \( x \in \mathbb{R}^n \) and \( r > 0\), by \( B(x, r) \) we denote the open ball in \( \mathbb{R}^n\) with radius \( r \) centered in \( x\), i.e. \( B(x, r) = \{ y \in \mathbb{R}^n \mid d(y, x) < r \} \) (\( d \) is the Euclidean metric on \( \mathbb{R}^n\)). By \( \overline{B}(x, r) \) we will denote the corresponding closed ball. It is not hard to prove the following proposition (see \([11]\)).

**Proposition 2.1**

1. Let \( x \in \mathbb{Q}^n\) and \( r \in \mathbb{Q}, r > 0\). Then \( \overline{B}(x, r) \) is a co-c.e. set in \( \mathbb{R}^n\).
2. Suppose \( S \) and \( T \) are co-c.e. sets in \( \mathbb{R}^n\). Then \( S \cap T \) is a co-c.e. set.

Let \( n, N \in \mathbb{N}, N \geq 1 \). Let \( a_0, \ldots, a_n \) be geometrically independent points in \( \mathbb{R}^N\) (i.e. points such that \( a_1 - a_0, \ldots, a_n - a_0 \) are linearly independent vectors). Then the convex hull \( \sigma \) of the set \( \{a_0, \ldots, a_n\} \) is called \( n\) - simplex in \( \mathbb{R}^N\) spanned by \( a_0, \ldots, a_n\). We say that \( a_0, \ldots, a_n \) are vertices of \( \sigma \) and we say that \( n \) is the dimension of \( \sigma \). It is known that the set of vertices \( \{a_0, \ldots, a_n\} \) and the number \( n \) are uniquely determined by \( \sigma \) and that \( \sigma \) can be described as the set of all points \( x \in \mathbb{R}^N\) of the form

\[
x = \sum_{i=0}^{n} t_i a_i ,
\]

where \( t_0, \ldots, t_n \) are non-negative real numbers such that \( \sum_{i=0}^{n} t_i = 1\), see \([8]\).

If \( \sigma \) is a simplex with vertices \( a_0, \ldots, a_n \) and \( i_0, \ldots, i_k \) numbers such that \( 0 \leq i_0 < \cdots < i_k \leq n\), then the simplex spanned by the points \( a_{i_0}, \ldots, a_{i_k}\) is called a face of \( \sigma \). If \( \tau \) is a face of \( \sigma \) and \( \tau \neq \sigma\), then we say that \( \tau \) is a proper face of \( \sigma \). The union of all proper faces of \( \sigma \) is denoted by \( \text{Bd} \sigma \) and it is called the boundary of \( \sigma \). The set \( \sigma \setminus \text{Bd} \sigma \) is called the interior of \( \sigma \) and it is denoted by \( \text{Int} \sigma \). In general, \( \text{Bd} \sigma \) is not the topological boundary of \( \sigma \) in \( \mathbb{R}^N\) and \( \text{Int} \sigma \) is not the topological interior of \( \sigma \) in \( \mathbb{R}^N\).

A plane in \( \mathbb{R}^N\) is a subset of the form \( a + W\), where \( a \in \mathbb{R}^N\) and \( W \) is a vector subspace of \( \mathbb{R}^N\). Suppose \( a_0, \ldots, a_n \) are geometrically independent points in \( \mathbb{R}^N\). Let \( W \) be a vector subspace of \( \mathbb{R}^N\) generated by vectors \( a_1 - a_0, \ldots, a_n - a_0\) and let \( \pi = a_0 + W\). We say that \( \pi \) is the plane spanned by \( a_0, \ldots, a_n\). It is easy to see that

\[
\pi = \left\{ \sum_{i=0}^{n} t_i x_i \middle| t_0, \ldots, t_n \in \mathbb{R}, \sum_{i=0}^{n} t_i = 1 \right\}.
\]
Therefore $\sigma \subseteq \tau$, where $\sigma$ is the simplex spanned by $a_0, \ldots, a_n$.

Let $K$ be a nonempty finite family of simplices in $\mathbb{R}^N$. We say that $K$ is a simplicial complex in $\mathbb{R}^N$ if the following holds:

1. if $\sigma \in K$ and $\tau$ is a face of $\sigma$, then $\tau \in K$;
2. if $\sigma, \tau \in K$ and $\sigma \cap \tau \neq \emptyset$, then $\sigma \cap \tau$ is a face of $\sigma$ and $\tau$.

If $K$ is a simplicial complex, let

$$|K| = \bigcup_{\sigma \in K} \sigma.$$ 

A subset $P$ of $\mathbb{R}^N$ is said to be a polyhedron if there exists a simplicial complex $K$ such that $P = |K|$.

In the following proposition we state some elementary properties of simplicies and simplical complexes (see [8]).

**Proposition 2.2**

1. Let $\sigma$ be a simplex in $\mathbb{R}^N$. Then $\sigma$ is a compact set and $\operatorname{Int} \sigma$ is a dense set in $\sigma$. Suppose $a_0, \ldots, a_n$ are vertices of $\sigma$ and let $\pi$ be the plane spanned by these points. Then $\operatorname{Int} \sigma$ is an open set in $\pi$.
2. Let $K$ be a simplical complex in $\mathbb{R}^n$. Then for each $x \in |K|$ there exists unique $\sigma \in K$ such that $x \in \operatorname{Int} \sigma$.

**CO-C.E. POLYHEDRA**

In this section we prove that computable points in a co-c.e. polyhedron are dense. First we need the following simple lemma.

**Lemma 3.1** Let $\pi$ be a plane in $\mathbb{R}^N$, $x \in \mathbb{R}^N$ and $r > 0$ such that $d(x, \pi) < r$. Then $\overline{B}(x, r) \cap \pi$ is a closed ball in $\pi$.

**Proof.** Let $x_0$ be the orthogonal projection of $x$ to $\pi$, i.e. the point $x_0 \in \pi$ such that the vectors $x - x_0$ and $y - x_0$ are orthogonal for each $y \in \pi$. Then for each $y \in \pi$ we have

$$(d(x, y))^2 = (d(x, x_0))^2 + (d(x_0, y))^2. \tag{1}$$

Since $d(x, \pi) = d(x, x_0)$ the number $s = \sqrt{r^2 - (d(x, x_0)^2)}$ is positive. Let $\overline{B}_\pi(x_0, s)$ be the closed ball in $\pi$ (with respect to the Euclidean metric on $\pi$) centered in $x_0$ with radius $s$. It is easy to conclude from (1) that

$$\overline{B}(x, r) \cap \pi = \overline{B}_\pi(x_0, s).$$

**Lemma 3.2** Let $K$ be a simplicial complex in $\mathbb{R}^N$. Suppose $x \in |K|$ is a point with the following property: there exists unique $\sigma \in K$ such that $x \in \sigma$. Then for each $\varepsilon > 0$ there exist $a \in \mathbb{Q}^n$ and $v \in \mathbb{Q}$, $v > 0$ such that $d(x, a) < \varepsilon$ and $v < \varepsilon$ and such that the set $\overline{B}(a, v) \cap |K|$ is homeomorphic to the unit closed ball $B^n$ in $\mathbb{R}^n$ for some $n \in \mathbb{N}$.

**Proof.** Let $\varepsilon > 0$. Let $\sigma \in K$ be such that $x \in \sigma$. Let $a_0, \ldots, a_n$ be the vertices of $\sigma$ and let $\pi$ be the plane spanned by these points.

By the assumption of the lemma, for each $\tau \in K$ such that $\tau \neq \sigma$ we have $x \notin \tau$ and since $\tau$ is closed in $\mathbb{R}^N$ (by Proposition 2.2(ii)) there exists $r > 0$ such that $B(x, r) \subseteq \mathbb{R}^N \setminus \tau$. There are only finitely many simplices in $K$ and therefore there exists $r > 0$ such that
for each \( \tau \in K \) such that \( \tau \neq \sigma \). It follows that

\[
B(x, r) \cap \tau = \emptyset
\]

(2)

Note that \( x \in \text{Int} \sigma \). Otherwise we have \( x \in \text{Bd} \sigma \) and therefore there exists a proper face \( \tau \) of \( \sigma \) such that \( x \in \tau \). Hence \( x \in \tau \), \( x \in \sigma \) and \( \tau \neq \sigma \) which contradicts the assumption of the lemma. Since \( \text{Int} \sigma \) is open in \( \pi \) (by Proposition 2.2(i)), there exists \( s > 0 \) such that

\[
B(x, s) \cap \pi \subseteq \text{Int} \sigma
\]

and consequently

\[
B(x, s) \cap \pi = B(x, s) \cap \sigma.
\]

(3)

Let \( t = \min \{ r, s, \varepsilon \} \). It follows from (2) and (3) that

\[
B(x, t) \cap K = B(x, t) \cap \sigma \quad \text{and} \quad B(x, t) \cap \pi = B(x, t) \cap \sigma
\]

and therefore

\[
B(x, t) \cap K = B(x, t) \cap \pi.
\]

(4)

Let us choose \( a \in Q^N \) and \( v \in Q \), \( v > 0 \), such that \( d(x, a) < \frac{t}{4} \) and \( \frac{t}{4} < v < \frac{t}{2} \). Then

\[
d(a, \pi) \leq d(a, x) < \frac{t}{4} < v
\]

and it follows from Lemma 3.1 that \( \overline{B}(a, v) \cap \pi \) is a closed ball in \( \pi \). On the other hand, inequalities \( d(x, a) < \frac{t}{4} \) and \( v < \frac{t}{2} \) easily imply

\[
\overline{B}(a, v) \subseteq B(x, t).
\]

(5)

It follows from (4) and (5) that

\[
\overline{B}(a, v) \cap K = \overline{B}(a, v) \cap \pi.
\]

(6)

In general, if \( W \) is an \( n \)-dimensional vector subspace of \( R^N \), then \( W \) is isometrically homeomorphic to \( R^n \) (since \( W \) has orthonormal basis; if \( n = 0 \) we take \( R^0 = \{ 0 \} \)). The same holds for \( a + W \) for any \( a \in R^N \). Therefore \( \pi \) is isometrically homeomorphic to \( R^n \) and consequently each closed ball in \( P \) is isometrically homeomorphic to a closed ball in \( R^n \). Since each closed ball in \( R^n \) is homeomorphic to the unit closed ball \( B^n \) and \( \overline{B}(a, v) \cap \pi \) is a closed ball in \( \pi \), we have that \( \overline{B}(a, v) \cap \pi \) is homeomorphic to \( B^n \). Hence, by (6),

\[
\overline{B}(a, v) \cap K
\]

is homeomorphic to \( B^n \). Since \( t \leq \varepsilon \), we have \( d(x, a) < \varepsilon \) and \( v < \varepsilon \).

\[ \square \]

**Theorem 3.3** Let \( P \) be a polyhedron in \( R^N \). Suppose \( P \) is a co-c.e. set. Then the set of all computable points in \( P \) is dense in \( P \).

*Proof.* Let \( K \) be a simplicial complex in \( R^N \) such that \( P = |K| \). Let \( S \) be the set of all \( x \in P \) which have the following property: there exists unique \( \sigma \in K \) such that \( x \in \sigma \). Then \( S \) is a dense set in \( P \).
Namely, let \( y \in P \) and \( \varepsilon > 0 \). Let \( \sigma \in K \) be a simplex of largest dimension such that \( y \in \sigma \). Since \( \text{Int} \sigma \) is dense in \( \sigma \) (Proposition 2.2), there exists \( x \in \text{Int} \sigma \) such that

\[
d(y, x) < \varepsilon.
\]

(7)

Suppose \( x \in \tau \) for some \( \tau \in K \). Then

\[
x \in \sigma \cap \tau
\]

(8)

and it follows that \( \sigma \cap \tau \) is a face of \( \sigma \) and \( \tau \).

Suppose \( \sigma \cap \tau \) is a proper face of \( \sigma \). Then (8) implies \( x \in \text{Bd} \sigma \) which is impossible since \( x \in \text{Int} \sigma \). Hence \( \sigma \cap \tau = \sigma \) and it follows that \( \sigma \) is a face of \( \tau \).

Suppose \( \sigma \) is a proper face of \( \tau \). Then the dimension of \( \sigma \) is less than the dimension of \( \tau \) and this, together with \( y \in \sigma \subseteq \tau \), contradicts the fact that \( \sigma \) is a simplex in \( K \) of largest dimension such that \( y \in \sigma \). Therefore \( \sigma = \tau \) and this means that \( \sigma \) is a unique simplex in \( K \) which contains \( x \). Hence \( x \in S \) and this together with (7) proves that \( S \) is dense in \( P \).

Now we claim that for each \( x \in S \) and each \( \varepsilon > 0 \) there exists a computable point \( z \in P \) such that \( d(x, z) < \varepsilon \). This and the fact that \( S \) is dense in \( P \) will imply that the set of all computable points in \( P \) is dense in \( P \).

Let \( x \in S \) and \( \varepsilon > 0 \). By Lemma 3.2 there exist \( a \in \mathbb{Q}^n \) and \( v \in \mathbb{Q} \), \( v > 0 \), such that \( d(x, a) < \frac{\varepsilon}{2} \) and \( v < \frac{\varepsilon}{2} \) and such that \( \overline{B}(a, v) \cap P \) is homeomorphic to \( B^n \) for some \( n \in \mathbb{N} \). This and the fact that \( \overline{B}(a, v) \cap P \) is a co-c.e. set in \( \mathbb{R}^n \), which follows from Proposition 2.1, imply by [7] that \( \overline{B}(a, v) \cap P \) contains a computable point. Hence there exists a computable point \( z \in P \) such that \( z \in \overline{B}(a, v) \). Now \( d(x, a) < \frac{\varepsilon}{2} \) and \( d(a, z) \leq v < \frac{\varepsilon}{2} \)

give \( d(x, z) < \varepsilon \).

\[ \Box \]

REFERENCES


