Hyperconnectedness in Ideal Supra Topological Spaces

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Abstract

The aim of this paper is to introduce and study hyperconnectedness in ideal supra topological spaces (briefly $iS$ – hyperconnectedness). Characterizations and properties of $iS$ – hyperconnectedness are provided and preservation functions of $iS$ – hyperconnectedness are investigated.

Keywords

$iS$ – hyperconnectedness, $iS$ – dense set, $iS$ – nowhere dense set, ideal topological space.

SUBJECT CLASSIFICATION

54A05, 54A10, 54C08, 54C10.

INTRODUCTION

The notion of ideal in topological space was first introduced by Kuratowski[1] and Vaidyanathswamy[2]. Further properties of ideal topological spaces was investigated by Jankovic and Hamlett [3]. Applications to many fields were investigated in [4],[5],[6],[7],[8], etc. The concept of hyperconnectedness[9] or equivalently $D$- spaces[10], semi-connected spaces[11], s-connected spaces[12] and irreducible spaces[13] was investigated and studied in the literature. In 1983 A. S. Mashhour et al. [14] developed the notion of supra topological spaces and studied the concept of supra- continuity. Ideal on supra spaces is investigated by Modak and Mistry[15]. Ekici [16] introduced and studied hyperconnectedness in ideal topological spaces. In [17] further properties of ideal supra topological spaces are investigated. This paper aimed to introduce and study the concept of hyperconnectedness in ideal supra topological spaces, we named by $iS$ – hyperconnectedness. The notions of $iS$ – dense set, $iS$ – nowhere dense set are introduced. Properties and characterizations of $iS$ – hyperconnectedness in ideal supra topological space are provided and preservation functions are investigated.

1. Preliminaries

Throughout this paper $X$ and $Y$ will denote topological spaces which has no separation axioms, unless otherwise stated. The closure and interior of a subset $A$ in $X$ is denoted by $\text{cl}(A)$ and $\text{int}(A)$ respectively. A topological space $X$ is hyperconnected [9] if every pair of nonempty open sets of $X$ has nonempty intersection. Ideal on $X$ is defined in [15] and supra topological space $(X, S)$ is defined and studied in [14], we will denote it by $XS$. The members of $S$ are called supra-open ($S$-open) sets and its complement is supra-closed ($S$-closed). Let $(X, \tau)$ be a topological space. “The $S$ – interior and $S$ – closure of $A$ in $(X, S)$ are denoted as $\text{int}_S(A)$ and $\text{cl}_S(A)$ respectively [14].

**Theorem 2.1.** [14] In $XS$, if $A \subseteq X$. Then

1. $\text{int}_S(A) \subseteq A$.
2. $\text{int}_S(A) = A$ “if and only if” $A \in S$.
3. $A \subseteq \text{cl}_S(A)$.
4. $X \setminus \text{int}_S(A) \subseteq \text{cl}_S(X \setminus A)$
5. $\text{cl}_S(A) = A$ if and only if $A$ is supra closed.
6. $x \in \text{cl}_S(A)$ if and only if every supra-open set $U$ containing $x$, $U \cap A \neq \emptyset$.

An $XS$ with an ideal $I$ on $X$ is called an ideal supra topological space [15] or simply $XS_I$. 


Definition 2.2 [15]
For the space $XS_I$, the $S$-local function $(,)^*: P(X) \to P(X)$ of I on X with respect to S and I is given by

$$(A)^* (I, S) = \{ x \in X : U \cup A \not\in I, U \in S \} \text{ for } A \subseteq X \text{ and } S(x) = \{ U \in S : x \in U \}. \text{ For simplicity we use } A^*. \text{ }$$

Definition 2.3 [17]
For $XS_I$, the set operator $cl^*$ is called a $(*, S)$-supra closure and is defined as $cl^*(A) = A \bigcup A^* \text{ for } A \subseteq X$. The supra topology $S^{\tau}$ is finer than $S$, generated by $cl^*$. $S^{\tau} = \{ U \subseteq X : cl^*(X \cup U) = X \cup U \}$. For any ideal supra space $XS_I$, the collection $\{ U \setminus G : U \in S, G \in I \}$ is a base for $S^{\tau}$. The elements of $S^{\tau}$ are called $S^* - \text{open}$ and the complement of an $S^* - \text{open}$ set is called $S^* - \text{closed}$. Some properties of $S$-local function are given in the following.

Proposition 2.4 [5]
In $XS_I$, if $A, B$ be any subsets of $X$, then,

1. $A \subseteq B \Rightarrow A^* \subseteq B^*$.
2. $A^* = cl^*_S(A) \subseteq cl^*_S(A)$. 

Proposition 2.5
For $XS_I$, if $A, B$ are any subsets of $X$ then,

1. $A \subseteq cl^* (A)$. [17]
2. If $A \subseteq B$ then $cl^*(A) \subseteq cl^*(B)$.

Proof.
(2): Follows from Proposition 2.4 and the definition of $cl^*$.

Definition 2.6
If $A$ is a subset of $XS$, then $A$ is $S$-pre-[18] (resp. $S$-semi-[19], $S$-b-[20], $S$-$\beta$-[21], $S$-regular-[19]) open, briefly $p_A = (\text{resp. } s_A, s_A, \beta_A, r_A)$ open, if $A \subseteq \text{int}_S(cl^*_S(A)) \text{ (resp. } A \subseteq \text{cl}^*_S(\text{int}_S(A)), A \subseteq \text{cl}^*_S(\text{int}_S(A)) \bigcup \text{int}_S(\text{cl}^*_S(A)), A \subseteq \bigcup \text{int}_S(\text{cl}^*_S(A)) \text{, } A \subseteq \text{int}_S(\text{cl}^*_S(A)) \text{).}$

Definition 2.7
For $XS_I$ and $A \subseteq X$, $A$ is an

$S$-$I^*$ (resp. $S$-pre-$I^*$, $S$-semi-$I^*$, $S$-b-$I^*$, $S$-$\beta$-$I^*$) open set, briefly $SIO$ (resp. $p_I, s_I, \beta_I, r_I$) if $A \subseteq \text{int}_S(A)$ $\bigcup \text{int}_S(A) \subseteq \text{cl}^*_S(A)$ (resp. $A \subseteq \text{int}_S(\text{cl}^*_S(A))$, $A \subseteq \text{cl}^*_S(\text{int}_S(I)) \bigcup \text{int}_S(\text{cl}^*_S(I))$, $A \subseteq \text{cl}^*_S(\text{int}_S(I)) \bigcup \text{int}_S(\text{cl}^*_S(I))$). The complements of the above mentioned sets are called their respective closed sets and denoted by $Slc$ (resp. $p_I, s_I, \beta_I, r_I$).

Remark 2.8
$S$-openness and $Slc$ are independent as illustrate in the next example.

Example 2.9
Let $X=\{a, b, c\}$, $S=\{X, \emptyset, \{a, b, c\}\}$, $I=\{\emptyset, \{b\}\}$. Then, $\{a, b\}$ is $S$-open but not $Slc$ and (c) is $Slc$ but not $S$-open.

Proposition 2.10
For $A \subseteq X$ where $X$ is an $XS_I$, then each $p_I, r_I$ (resp. $s_I, b_I, \beta_I, r_I$) set is $p_I, r_I$ (resp. $s_I, b_I, \beta_I, r_I$) open.

Proof.
Suppose that $A$ is a $p_I$ set. Hence $A \subseteq \text{int}_S(\text{cl}^*_S(A)) \subseteq \text{int}_S(\text{cl}^*_S(A))$. Hence $A$ is $p_I$ open.
We can follow the same method to demonstrate the remaining cases.

Proposition 2.11
In $XS_i$ "The following implications" are true for any $A \subset X$.

1. $p_i I_o \Rightarrow b_i I_o \Rightarrow \beta_i I_o$.
2. $s_i I_o \Rightarrow b_i I_o \Rightarrow \beta_i I_o$.

Proof.
This is obvious from Definition 2.7.

3. *Hyperconnected Spaces.*

Definition 3.1
For $A \subset X$ where $X$ is $XS_i$, then $A$ is

1. $iS$ - dense if $cl^*(A) = X$.
2. $iS$ - nowhere dense if $int_i(cl^*(A)) = \phi$.

Definition 3.2. [22] An $XS$ space is,

1. $S$- hyperconnected if each $S$- open set $A \neq \phi$ of $X$ is $S$- dense, that is $cl_s(A) = X$.
2. $S$- connected if $X \neq A \cup B$ where $A$ (resp. $B$) is nonempty $S$- open sets of $X$.

Definition 3.3
An $XS_i$ is,

1. $iS$ - hyperconnected if any $S$- open set $A \neq \phi$ is $iS$ - dense.
2. $iS$ - connected if $X \neq A \cup B$ where $A$ (resp. $B$) is a nonempty $S$- (resp. $S^*$) open sets of $X$.

Remark 3.4
From Definitions 3.3 and the facts that every $S$- hyperconnected is $S$- connected, we have:

\[ XS_i \text{ is } iS \text{- hyperconnected } \Rightarrow XS_i \text{ is } S \text{- hyperconnected} \]

\[ \Rightarrow \]

\[ XS_i \text{ is } iS \text{- connected } \Rightarrow XS_i \text{ is } S \text{- connected} \]

Reverse trends are generally incorrect as shown below.

Example 3.5
Let $X= \{a, b, c\}$, $S= \{X, \phi, \{a\}, \{a, b\}, \{a, c\}\}$ and $I= \{\phi, \{b\}\}$. Then $XS_i$ is $S$ - hyperconnected but $XS_i$ is not $iS$ – connected.

Example 3.6
Let $X= \{a, b, c\}$, $S= \{X, \phi, \{a\}, \{a, b\}, \{b, c\}\}$ and $I= \{\phi, \{a\}\}$. Then $XS_i$ is $iS$ – connected but $XS$ is not $S$ - hyperconnected.

Definition 3.7
If $A$ is a subset of $XS_i$, then $A$ is called supra semi-1-open (briefly $s_i I_o$) if $A \subset cl_s(int^*(A))$ and its complement is called semi-1-closed (briefly $s_i I_c$)
Proof.
If $A$ is an $\mathcal{S}_I$ set. Then $A \subseteq cl^\prec (\text{int}_s (A)) \subseteq cl^\prec (\text{int}^* (A))$. Hence $A$ is $\mathcal{S}_I$.

The reverse direction is generally incorrect as shown below.

Example 3.9
Let $X= \{a, b, c, d\}$, $S = \{X, \emptyset, \{a, b\}, \{b, c\}, \{a, b, c\}\}$ and $I = \{\emptyset, \{c\}\}$. Then $\{b, d\}$ is $\mathcal{S}_I$ but not $\mathcal{S}_I$.

Lemma 3.10
A subset $A$ of $XS_I$ is $\mathcal{S}_I$ if and only if there exists $U \in S$ such that $U \subseteq A \subseteq cl^\prec (U)$.

Proof.
If $A$ is $\mathcal{S}_I$. Then $A \subseteq cl^\prec (\text{int}_s (A))$. Put $U = \text{int}_s (A)$. Then $U \subseteq A \subseteq cl^\prec (U)$.

Conversely, let $U \subseteq A \subseteq cl^\prec (U)$ for $U \in S$. Since $U \subseteq A$, then $U \subseteq \text{int}_s (A)$. Therefore $cl^\prec (U) \subseteq cl^\prec (\text{int}_s (A))$ and hence $A \subseteq cl^\prec (\text{int}_s (A))$. Thus $A$ is $\mathcal{S}_I$.

Lemma 3.11
A subset $A$ of $XS_I$ is $\mathcal{S}_I$ if and only if there exists $U \in S \tau$, such that $U \subseteq A \subseteq cl^\prec (U)$.

Proof.
Similar to that of Lemma 3.10.

Theorem 3.12
In $XS_I$, the following are equivalent:

(a) $X$ is $\mathcal{S}_I$—hyperconnected.

(b) $A$ is $\mathcal{S}_I$—dense or $\mathcal{S}_I$—nowhere dense, for every subset $A$ of $X$.

(c) $A \cap B \neq \emptyset$ for any $S$-open set $\emptyset \neq A \subseteq X$ and any $S^*$-open set $\emptyset \neq B \subseteq X$.

(d) $A \cap B \neq \emptyset$, where $\emptyset \neq A \subseteq X$ is any $\mathcal{S}_I$ set and $\emptyset \neq B \subseteq X$ is any $\mathcal{S}_I$ set.

Proof.
(a) $\Rightarrow$ (b): Let $X$ be $\mathcal{S}_I$—hyperconnected. Suppose that the subset $A$ is not $\mathcal{S}_I$—nowhere dense. Then $cl^\prec (X \setminus cl^\prec (A)) = \text{int}_s (cl^\prec (A)) \neq X$. Since $\text{int}_s (cl^\prec (A)) \neq \emptyset$, so by (a), $cl^\prec (\text{int}_s (cl^\prec (A))) = X$. Since $cl^\prec (\text{int}_s (cl^\prec (A))) = X \subseteq cl^\prec (A)$, then $cl^\prec (A) = X$. Hence $A$ is $\mathcal{S}_I$—dense.

(b) $\Rightarrow$ (c): "Suppose" that $A \subseteq S$ and $B \subseteq S \tau$ are disjoint. Then $cl^\prec (A) \cap B = \emptyset$ and $A$ is not $\mathcal{S}_I$—dense. Since $\emptyset \neq A \subseteq S$, so $A \subseteq \text{int}_s (cl^\prec (A))$ and $A$ is not $\mathcal{S}_I$—"nowhere dense". This is a contradiction. Hence $A \cap B \neq \emptyset$.

(c) $\Rightarrow$ (d): Suppose that $\emptyset \neq A \subseteq S$ and $\emptyset \neq B$ are disjoint where $A$ is $\mathcal{S}_I$ set and $B$ is $\mathcal{S}_I$ set. By Lemmas (3.10) and (3.11), there are $H \subseteq S$ and $M \subseteq S \tau$, such that $H \subseteq A \subseteq cl^\prec (H)$ and $M \subseteq A \subseteq cl^\prec (M)$. But $A$ and $B$ are nonempty, hence $H$ and $M$ are nonempty. But $H \cap M \subseteq A \cap B = \emptyset$. This is a contradiction.

(d) $\Rightarrow$ (a): Let the intersection of $A$ and $B$ be empty, where $A$ is any nonempty $\mathcal{S}_I$ set and $B$ is any nonempty $\mathcal{S}_I$ set. Since every $S$-open set is $\mathcal{S}_I$ and every $S^*$-open set is $\mathcal{S}_I$. Then $X$ is $\mathcal{S}_I$—hyperconnected.

Definition 3.13
The supra semi"$-I-cl$ (resp., $S-semi-I-cl$, $S-pre-I-cl$, $S-\beta-I-cl$) of a subset $A$ of $XS_I$, symbolized it as, $\mathcal{S}_I cl (\text{resp. } s_I cl, p_I cl, \beta_I cl)$ of $A$, is the intersection of all $s_I cl$ (resp. $s_I cl$, $p_I cl$, $\beta_I cl$) sets of $X$ containing $A$. 
Lemma 3.14
For a subset K of XSt, we have

(1) \( s_i Icl(K) = K \cup int_s(cl^*(K)) \).
(2) \( s_Icl(K) = K \cup int^*(cl_s(K)) \).
(3) \( p_s Icl(K) = K \cup cl_s(int^*(K)) \).
(4) \( \beta_s Icl(K) = K \cup int^*(cl_s(int^*(K))) \).

Proof.
(4) The proof is similar to that of Lemma 13[16]. That is, since \( \beta_s Icl(K) \) is \( \beta_s Ic \), then, \( \beta_s Icl(K) \) is \( \beta_s Ic \) containing \( K \). S \( \beta_s Icl(K) \) is \( \beta_s Ic \) containing \( K \).
(5) \( \beta_s Icl(K) = K \cup int^*(cl_s(int^*(K))) \).

Similarly we can prove (1), (2) and (3).

Theorem 3.15
In XSt the following are equivalent:

(1) \( X \) is \( iS - hyperconnected \).
(2) \( A \) is \( iS - dense \) for any \( \beta_s Io \) set \( \phi \neq A \subset X \).
(3) \( A \) is \( iS - dense \) for any \( b_s Io \) set \( \phi \neq A \subset X \).
(4) \( A \) is \( iS - dense \) for any \( p_s Io \) set \( \phi \neq A \subset X \).
(5) \( s_s Icl(A) = X \) for any \( p_s Io \) set \( \phi \neq A \subset X \).
(6) \( p_s Icl(A) = X \) for any \( s_s Io \) set \( \phi \neq A \subset X \).

Proof. (1) \( \Rightarrow \) (2): Suppose that \( A \) is a nonempty \( \beta_s Io \) subset of \( X \). Then \( int_s(cl^*(A)) \neq \phi \). So \( X = cl^*(int_s(cl^*(A))) = cl^*(A) \).

(2) \( \Rightarrow \) (3): Since every \( b_s Io \) set is \( \beta_s Io \). So (3) hold.

(3) \( \Rightarrow \) (4): Since every \( p_s Io \) is \( b_s Io \). So (4) hold.

(4) \( \Rightarrow \) (5): Let \( A \) be a nonempty \( p_s Io \) subset of \( X \). Then by Lemma 3.14,

\[ s_s Icl(A) = A \cup int_s(cl^*(A)) = A \cup int_s(X) = X. \]

(5) \( \Rightarrow \) (6): "Let \( A \) be a nonempty" \( s_s Io \) subset of \( X \). Then by (5) and Lemma 3.14,

\[ X = s_s Icl(int_s(A)) = int_s(A) \cup int_s(cl^*(int_s(A))) \subset A \cup cl^*(int_s(A)) \subset A \cup cl_s(int^*(A)) \subset p_s Icl(A). \]

Hence \( p_s Icl(A) = X. \)

(6) \( \Rightarrow \) (1): Can be proved similarly.
4. Preservation Theorems of \( iS - \) hyperconnectedness

**Definition 4.1**
The supra- semi- \( I \) - interior of a subset \( A \) of \( XS_t \) (briefly \( S_j li(A) \)) is the union of all \( s_j Io \) sets of \( X \) included in \( A \).

**Definition 4.2**
(1) A "function" \( f : XS_t \rightarrow YS_2 \) is \( s_j I \) - continuous if for \( \forall \in S_2 \), \( f^{-1}(V) \) is an \( s_j Io \) set in \( XS_t \).

(2) A "function" \( f : XS_t \rightarrow YS_2 \) is said to be \( r_j I \) - continuous if for nonempty \( r_j \) - open set \( V \) of \( Y \), if \( f^{-1}(V) \neq \phi \) then \( s_j li (f^{-1}(V)) \neq \phi \).

**Theorem 4.3**
\( f : XS_t \rightarrow YS_2 \) is \( s_j I \) - continuous function \( \Rightarrow f \) is \( r_j I \) - continuous.

**Proof.**
If \( A \) be any \( r_j \) - open subset of \( Y \) such that \( f^{-1}(V) \neq \phi \). So \( f^{-1}(V) \) is a nonempty \( s_j Io \) in \( X \). Hence \( f^{-1}(V) = s_j li (f^{-1}(V)) \neq \phi \). Thus \( f \) is \( r_j I \) - continuous.

**Theorem 4.4**
If \( f : XS_t \rightarrow YS_2 \) is an onto \( r_j I \) - continuous and \( XS_t \) is \( iS \) - hyperconnected, then \( YS_2 \) is \( S \) - hyperconnected.

**Proof.**
If \( Y \) is not \( S \) - hyperconnected then there are disjoint nonempty supra open sets \( A \) and \( B \) in \( Y \) [22]. Put \( U = int_j (cl_j(A)) \) and \( V = int_j (cl_j(B)) \). Hence \( U = int_j (cl_j(U)) \) and \( V = int_j (cl_j(V)) \). Thus \( U \) and \( V \) are disjoint nonempty \( r_j \) - open sets. Hence \( s_j li(f^{-1}(U)) \cap s_j li(f^{-1}(V)) \subseteq f^{-1}(U) \cap f^{-1}(V) = \phi \). Since \( f \) is an \( r_j I \) - continuous onto, then \( s_j li(f^{-1}(U)) \neq \phi \) and \( s_j li(f^{-1}(V)) \neq \phi \). Hence, by Lemma 3.8 and Theorem 3.12, \( X \) is non \( iS \) - hyperconnected. This contradicts the assumption.

By Theorem 4.3 and Theorem 4.4 we get,

**Corollary 4.5**
If \( f : XS_t \rightarrow YS_2 \) is an \( s_j I \) - continuous onto and \( XS_t \) is \( iS \) - hyperconnected, then \( YS_2 \) is \( S \) - hyperconnected.

**Definition 4.6.** \( f : XS_t \rightarrow YS_2 \) is \( S \) - continuous function \( f^{-1}(V) \in S_j \) for all \( V \in S_2 \).

**Remark 4.7**
\( S \) - continuous \( \rightarrow s_j I \) - continuous. The reverse is incorrect as shown below.

**Example 4.8**
Let \( X = \{a, b, c, d\} \) and \( S_1 = S_2 = \{X, \phi, \{b\}, \{d\}, \{b, d\}\} \) and \( f : XS_t \rightarrow YS_2 \) defined as \( f(a) = a, f(b) = d, f(c) = d, f(d) = b \).

So \( f \) is \( s_j I \) - continuous but not \( S \) - continuous.

By Theorem 4.3 and Remark 4.7 we have the following,
Remark 4.9.
S-continuous $\Rightarrow s \mathcal{I}$ -continuous $\Rightarrow r \mathcal{I}$ - continuous.

By Theorem 4.4 and Remark 4.9, we get,

Corollary 4.10.
If $f : XS \mathcal{I} \to YS_2$ is onto S-continuous and $XS \mathcal{I}$ is $iS$ - hyperconnected, then $YS_2$ is S- hyperconnected.

REFERENCES