Asymptotic Behavior of Solutions of Second Order Neutral Delay Difference Equations with “Maxima”

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Abstract

In this paper, we study the asymptotic behavior of solutions of the second order neutral delay difference equation with “maxima” of the form

\[\Delta\left(a_n\Delta(x_n + p_n x_{n-\tau})\right) + q_n \max_{[n-\sigma,n]} x_s = 0, \quad n \in \mathbb{N}(n_0)\]

Example are given to illustrate the main result.

Indexing terms/Keywords

Second order, asymptotic behavior, neutral, delay difference equations with “maxima”.

SUBJECT CLASSIFICATION

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INTRODUCTION

In this paper, we study the asymptotic behavior of solutions of the second order neutral delay difference equation with “maxima” of the form

\[\Delta\left(a_n\Delta(x_n + p_n x_{n-\tau})\right) + q_n \max_{[n-\sigma,n]} x_s = 0, \quad n \in \mathbb{N}(n_0)\]

where \(\Delta\) is the forward difference operator defined by \(\Delta x_n = x_{n+1} - x_n\) and \(n_0\) is a nonnegative integer subject to the following conditions:

- \(\tau\) and \(\sigma\) are positive integers;
- \(\{p_n\}\) and \(\{q_n\}\) are real sequences;
- \(\{a_n\}\) is a positive real sequence.

Let \(\theta = \max\{\tau, \sigma\}\). By a solution of equation (1.1), we mean a real sequence \(\{x_n\}\) satisfying equation (1.1) for all \(n \geq n_0 - \theta\). Such a solution is said to be oscillatory if it is neither eventually positive nor eventually negative and nonoscillatory otherwise. A nonoscillatory solution \(\{x_n\}\) of equation (1.1) is said to be weakly oscillatory if \(\Delta x_n\) is oscillatory. Equation (1.1) is said to be almost oscillatory if every solution \(\{x_n\}\) is either oscillatory or satisfies \(\lim_{n \to \infty} x_n = 0\).

In [1, 2, 3, 4, 8, 9, 13, 14, 15, 16, 17, 18], the authors investigated the oscillatory and asymptotic behavior of solutions of second order neutral difference equations without maxima. But very few results available in the literature dealing with oscillatory and asymptotic behavior of solution of second order neutral difference equations with “maxima”, see [5, 6, 7, 10, 11, 12] and the references cited therein.

The aim of this paper is to consider the cases \(q \geq 0\) and \(q\) changes the sign for all large \(n\), to give sufficient conditions for the existence and asymptotic nature of nonoscillatory solution of equation (1.1) with respect to their asymptotic behavior, all the solutions of equation (1.1) with \(M^+ = \{x_n \text{ is a solution of (1.1): } x_n\Delta x_n \geq 0 \text{ for all large } n\}\)

In Section 2, we establish the sufficient condition for the existence and asymptotic behavior of solutions of equation (1.1). In Section 3, we present some examples to illustrate the main results. The results presented in this paper are new and complement to the results reported in the literature for difference equation.
2 Main Results

In this section, we provide sufficient condition for the existence and asymptotic behavior of solution of equation (1.1) in $\mathbb{M}^+$. 

**Theorem 2.1** Assume that

$$p_n \geq 0 \text{ and nondecreasing for all } n \geq n_0$$

and

$$\lim_{n \to \infty} \sup_{n=n_0} \sum_{s=n_0}^{n-1} q_s = \infty$$

hold. Then for equation (1.1) we have $\mathbb{M}^+ = \emptyset$.

**Proof.** Assume that equation (1.1) has a solution $\{x_n\}$ in $\mathbb{M}^+$. Without loss of generality we may assume that $x_n > 0, \Delta x_n \geq 0, x_{n-1} > 0$ and $\Delta x_{n-1} \geq 0$ for all $n \geq n_1 \geq n_0$ (The proof is similar if $x_n < 0, x_{n-\sigma} < 0$ for all large $n$). Let

$$z_n = x_n + p_n x_{n-\tau}.$$ 

Then $z_n > 0$ for all $n \geq n_1$ and

$$\Delta z_n = \Delta x_n + \Delta (p_n x_{n-\tau}) = \Delta x_n + x_{n-\tau} \Delta p_n + p_{n+1} \Delta x_{n-\tau}.$$ 

By condition (2.1), we have $\Delta z_n > 0$ for all $n \geq n_1$. Suppose $x_n \in \mathbb{M}^+$, we have $\max_{[n-\sigma,n]} x_s = x_n$ and the equation (1.1) becomes

$$\Delta(a_n \Delta z_n) = -q_n x_n \leq 0 \text{ for } n \geq n_1.$$ 

Now,

$$\Delta \left( \frac{c_n \Delta x_n}{x_n} \right) = \frac{\Delta (c_n \Delta x_n)}{x_n} - \frac{c_{n+1} \Delta x_{n+1}}{x_{n+1}} \Delta x_n.$$ 

Since $\Delta z_n > 0$ and $\Delta x_n \geq 0$. We have by (2.4) and (2.5),

$$\Delta \left( \frac{c_n \Delta x_n}{x_n} \right) \leq -q_n.$$ 

Summing the last inequality from $n_1$ to $n-1$ we obtain

$$\frac{c_n \Delta x_n}{x_n} - \frac{c_{n_1} \Delta x_{n_1}}{x_{n_1}} \leq -\sum_{s=n_1}^{n-1} q_s.$$ 

Letting $n \to \infty$ and taking sup, we obtain $\Delta z_n < 0$ by (2.2) for all $n \geq n_1$, which is a contradiction. This completes the proof.

In the next theorem we consider the existence of solution in $\mathbb{M}^+$ when

$$-1 \leq p_1 \leq p_n \leq 0,$$

$$q_n \geq 0 \text{ for all } n \geq n_0,$$

and

$$\lim_{n \to \infty} \sum_{s=n_0}^{n} q_s = \infty$$

and

$$\lim_{n \to \infty} \sum_{s=n_0}^{n} \frac{1}{a_s} = \infty.$$
Theorem 2.2 Assume that (2.7)-(2.10) hold. Then for equation (1.1) we have $M^+ = \phi$.

Proof. Proceeding as in the proof of Theorem 2.1, we have (2.3) and (2.4). Since $x_n \in M^+$, we have

$$z_n \geq (1 + p_n)x_{n-1} > 0.$$ 

From (2.4), we have $a_n \Delta z_n$ is of one sign for all $n \geq n_1 \geq n_0$. If $a_n \Delta z_n < 0$ for $n \geq n_1$, then

$$a_n (\Delta z_n) \leq a_{n_1} (\Delta z_{n_1}) < 0 \text{ for all } n \geq n_1.$$

Dividing the last inequality by $a_n$ and then summing the resulting inequality from $n_1$ to $n - 1$, we obtain

$$z_n < z_{n_1} + a_{n_1} (\Delta z_{n_1}) \frac{1}{a_z}.$$ 

Letting $n \to \infty$ in the above inequality we obtain $z_n \to +\infty$ as $n \to \infty$, which is a contradiction. Here $\Delta z_n > 0$ for $n \geq n_1$. Now proceeding as in the proof of Theorem 2.1, we obtain by using the (2.9) that $\Delta z_n < 0$ for $n \leq n_1$, which is a contradiction. This completes the proof.

Finally, we examine the asymptotic behavior of solutions in $M^+$.

Theorem 2.3 Assume that condition (2.1) holds with $\{P_n\}$ is bounded. If

$$\limsup_{n \to \infty} \sum_{s=n_1}^{n-1} q_s \left( \frac{1}{a_s} \right) = \infty, n_1 \geq n_0,$$  \hspace{1cm} (2.11)

then every solution of equation (1.1) in the class $M^+$ is unbounded.

Proof. Let $\{x_n\}$ be a solution of equation (1.1) such that $x_n \in M^+$. Then proceeding as in the proof of Theorem 2.1, we have $z_n > 0, \Delta z_n > 0$ for all $n \geq n_1$. Define

$$w_n = -a_n \Delta x_n \sum_{s=n_1}^{n-1} \frac{1}{a_s}.$$

then

$$\Delta w_n = q_n \sum_{s=n_1}^{n-1} \frac{1}{a_s} \Delta x_n - \frac{a_n \Delta x_n}{x_n} + \frac{a_{n+1} \Delta x_{n+1}}{x_n x_{n+1}} \Delta x_n \sum_{s=n_1}^{n-1} \frac{1}{a_s}.$$

$$= q_n \sum_{s=n_1}^{n-1} \frac{1}{a_s} \Delta x_n + \frac{a_{n+1} \Delta x_{n+1}}{x_n x_{n+1}} \Delta x_n \sum_{s=n_1}^{n-1} \frac{1}{a_s}.$$

$$\geq q_n \sum_{s=n_1}^{n-1} \frac{1}{a_s} \Delta x_n.$$

Summing the last inequality from $n_1$ to $n - 1$, we obtain

$$w_n - w_{n_1} \geq \sum_{s=n_1}^{n-1} q_s \left( \frac{1}{a_s} \right) - \sum_{s=n_1}^{n-1} \frac{\Delta x_s}{x_s}.$$  \hspace{1cm} (2.12)

As in the function $\frac{\Delta x_n}{x_n}$ is positive for $n \geq n_1$ then $\lim_{n \to \infty} \sum_{s=n_1}^{n-1} \frac{\Delta x_s}{x_s}$ exists. Assume that

$$\lim_{n \to \infty} \sum_{s=n_1}^{n-1} \frac{\Delta x_s}{x_s} = k < \infty.$$ 

Taking into account (2.11), and (2.12) we obtain $\lim_{n \to \infty} \sup w_n = \infty$, which gives a contradiction, since $w_n$ is negative for all $n \geq n_1$. Thus

$$\sum_{s=n_1}^{n-1} \frac{\Delta x_s}{x_s} < \infty.$$  \hspace{1cm} (2.13)

Now for all values of $n \geq n_1$ and $x \in M^+$, we have $x_n \geq x_{n_1} = c$, and consequently

$$\sum_{s=n_1}^{n-1} \frac{\Delta x_s}{x_s} \leq \frac{1}{c} \sum_{s=n_1}^{n-1} \Delta x_s \leq \frac{1}{c} (z_n - z_{n_1}).$$

From (2.13) we obtain
\[ \lim_{n \to \infty} x_n = \infty. \quad (2.14) \]

Since \( z_n = x_n + p_n x_{n-1} \) and \( x_n \) is nondecreasing we have \( z_n \leq (1 + p_n)x_n \). Thus from (2.14), by taking into account the boundedness of \( p_n \), we obtain \( \lim_{n \to \infty} x_n = \infty \). This completes the proof.

### 3 Examples

In this section we present some examples to illustrate the main results.

**Example 3.1** Consider the neutral difference equation of the form

\[ \Delta \left( \frac{1}{4^n} \Delta(x_n - \frac{1}{2}x_{n-2}) \right) + \frac{7}{4^{n+2}} \max_{[n-2, n]} x_2 = 0, n \geq 1. \quad (3.1) \]

Here \( \alpha_n = \frac{1}{4^n}, p_n = \frac{1}{2} \) and \( q_n = \frac{7}{4^{n+2}} \). It is easy to see that all conditions of Theorem 2.2 are satisfied except condition (2.9). Hence for equation (3.1) the class \( M^+ \) is not empty. In fact \( \{ x_n \} = \{ 2^n \} \) is one such solution belongs to the class \( M^+ \) of equation (3.1).

**Example 3.2** Consider the neutral difference equation of the form

\[ \Delta \left( \frac{1}{4^n} \Delta(x_n + 2x_{n-1}) \right) + \frac{1}{4^n} \max_{[n-3, n]} x_2 = 0, n \geq 1. \quad (3.2) \]

Here \( \alpha_n = \frac{1}{4^n}, p_n = 2 \), and \( q_n = \frac{1}{4^n} \). It is easy to see that all conditions of Theorem 2.3 are satisfied and hence every solution of equation (3.2) belongs to the class \( M^+ \) is unbounded. In fact \( \{ x_n \} = \{ 2^n \} \) is one such solution of equation (3.2) belongs to the class \( M^+ \).

### REFERENCES


