Fuzzy Soft Pretopological Spaces
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Abstract
In this paper, we define the concepts of a fuzzy soft pretopological space, a fuzzy soft interior function, a fuzzy soft pre-open set, a fuzzy soft pre-closed set, the trace of a fuzzy soft pretopology and study some of its properties. Also the fuzzy soft preneighbourhood system at a soft point, the degree of soft non-vacuity and the soft $\alpha$-cut are defined and fuzzy soft pretopologies were generated by fuzzy soft preneighbourhoods.

Keywords: Fuzzy soft set; fuzzy soft topology; fuzzy soft pretopology; fuzzy soft interior function and fuzzy soft preneighbourhood system.

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1. INTRODUCTION

Many fields of sciences deal with the uncertain data that may not be successfully modeled by the classical mathematics. There are some mathematical tools for dealing with uncertainties; two of them are fuzzy set theory, developed by Zadeh [11] and soft set theory, introduced by Molodtsov [6]. Maji et al. [5] introduced the concept of fuzzy soft set and some properties. Tany and Kandemir [9] introduced the definition of fuzzy soft topology. M. Brissaud [3] introduced the concept of a pretopology. In [2], R. Badard introduced the concept of a fuzzy pretopology.

In the present paper, we define the fuzzy soft pretopology, the fuzzy soft interior function with its properties, the fuzzy soft pre-open set, the fuzzy soft pre-closed set with some propositions and the trace of a fuzzy soft pretopology. Also, we give the definition of the fuzzy soft preneighbourhood system at a soft point, the definition of a degree of soft non-vacuity, soft $\alpha$-cut of a fuzzy soft subset. Finally, we generate fuzzy soft pretopologies by fuzzy soft preneighbourhoods and give some propositions.

2. PRELIMINARIES

Definition 2.1 [6]
Let $X$ be an initial universe set and $E$ be a set of parameters. A pair $(F, A)$, denoted by $F_A$, is called a soft set over $X$, where $F$ is a mapping given by $F : A \to P(X)$ such that $F(a) = \emptyset$ if $a \notin A$ and $F(a) \neq \emptyset$ if $a \in A$, where $\phi$ stands for the empty set and $P(X)$ is the set of all subsets of $X$.

In order to efficiently discuss usually soft sets over a universe $X$ are considered with a fixed set of parameters $E$. The family of these soft sets are denoted by $SS(X, E)$.

Definition 2.2 [7]
A soft set $F_A$ over $X$ is said to be a soft point if there is exactly one $a \in A$ such that $F(a) = \{x\}$ for some $x \in X$ and $F(a') = \emptyset$, for every $a' \in A \setminus \{a\}$. It will be denoted by $x_a$.

Definition 2.3 [11]
A soft point $x_a$ is said to belongs to a soft set $G_A$ if $a \in A$ and $F(a) = \{x\} \subseteq G(a)$. We write $x_a \in G_A$.

Definition 2.4 [11]
A fuzzy set $A$ in any arbitrary set $X$ is defined by the mapping $\mu_a : X \to [0, 1] = I$, where $\mu_a(x)$ states the degree of membership of $x$ in $A$. That is, a fuzzy set $A$ in $X$ can be presented by the set of ordered pairs $A = \{(x, \mu_a(x)) : x \in X\}$. The family of all fuzzy sets in $X$ is denoted by $I^X$.

Definition 2.5 [5]
Let $X$ be an initial universe set, $E$ be a set of parameters, $I^X$ denotes the set of all fuzzy sets of $X$ and $A \subseteq E$. A pair $(F, A)$, denoted by $F_A$, is called a fuzzy soft set over $X$ if $F$ is a mapping given by $F : A \to I^X$ such that $F(a) = 0_x$ if $a \notin A$ and $F(a) \neq 0_x$ if $a \in A$, where $0_x(x) = 0$ for all $x \in X$. Thus a fuzzy soft set $F_A$ over $X$ can be
represented by the set of ordered pairs \( F_A = \{ (e, F_A) : e \in A, F_A \subseteq I^X \} \). In other words, the fuzzy soft set is a parameterized family of fuzzy subsets of the set \( X \).

The set of all fuzzy soft sets over an initial universe \( X \) and a set of parameters \( E \) is denoted by \( FSS(X, E) \).

**Definition 2.6 [4]**

A fuzzy soft set \( F_A \) over \( X \) is called a null fuzzy soft set, denoted by \( \hat{\phi} \), if \( F(e) = \emptyset \) for all \( e \in A \subseteq E \).

**Definition 2.7 [8]**

A fuzzy soft set \( F_A \) over \( X \) is said to be an absolute fuzzy soft set, denoted by \( \mathcal{X} \), if \( A = E \) and \( F(e) \subseteq G(e) \), for all \( e \in A \subseteq E \), where \( 1_X(x) = 1 \) for all \( x \in X \).

**Definition 2.8 [5]**

A fuzzy soft set \( F_A \) is said to be a fuzzy soft subset of a fuzzy soft set \( G_B \) over a common universe \( X \) if \( A \subseteq B \) and \( F(e) \subseteq G(e) \), for all \( e \in A \).

**Definition 2.9 [1]**

The intersection of two fuzzy soft sets \( F_A \) and \( G_B \) over a common universe \( X \) is the fuzzy soft set \( H_C \) where \( C = A \cap B \) and \( H(e) = F(e) \cap G(e) \), for all \( e \in C \), and we write \( H_c = F_A \cap G_B \).

In particular if \( A \cap B = \phi \) or \( F(e) \cap G(e) = \emptyset \), for every \( e \in A \cap B \) then \( H(e) = \emptyset \).

**Definition 2.10 [5]**

The union of two fuzzy soft sets \( F_A \) and \( G_B \) over a common universe \( X \) is the fuzzy soft set \( H_C \) where \( C = A \cup B \) and for all \( e \in C \), \( H(e) = F(e) \cap G(e) \), for all \( e \in A \cup B \), and in this case we write \( H_c = F_A \cup G_B \).

**Definition 2.11 [5]**

The fuzzy soft complement of a fuzzy soft set \( F_A \), denoted by \( F_A' \), and is defined as \( F_A = (F_A')' \), where \( F_A (e) = 1 - F_A (e) \), for every \( e \in A \). Clearly \( (F_A') = F_A' \) and \( (\mathcal{X})' = \mathcal{X} \).

**Definition 2.12 [9]**

A fuzzy soft topology \( \tau \) on \( (X, E) \) is a family of fuzzy soft sets over \( (X, E) \), satisfying the following properties:

1. \( \mathcal{X}, \hat{\phi} \in \tau \)
2. If \( F_A, G_B \in \tau \) then \( F_A \cap G_B \in \tau \)
3. If \( (F_A)_j \in \tau \), for every \( j \in J \) then \( \bigcup_{j \in J} (F_A)_j \in \tau \).

**Definition 2.13 [8]**

If \( \tau \) is a fuzzy soft topology on \( (X, E) \), the triple \( (X, E, \tau) \) is said to be a fuzzy soft topological space. Each member of \( \tau \) is called a fuzzy soft open set in \( (X, E, \tau) \). The complement of a fuzzy soft open set is called fuzzy soft closed.

**Definition 2.14 [9]**

Let \( (X, E, \tau) \) be a fuzzy soft topological space and \( F_A \) be a fuzzy soft set over \( (X, E) \). The fuzzy soft closure of \( F_A \), denoted by \( \bar{F}_A \), is the intersection of all fuzzy soft closed sets containing \( F_A \). That is, \( \bar{F}_A = (G_B : G_B \text{ is fuzzy soft closed and } F_A \subseteq G_B) \). Clearly, \( \bar{F}_A \) is the smallest fuzzy soft closed set over \( (X, E) \) which contain \( F_A \).
Theorm 2.15 [10]
Let \( c : \text{FSS}(X,E) \to \text{FSS}(X,E) \) be a function satisfying the following:

1. \((c)\) \( c(\phi) = \phi \);
2. \((c)\) \( c(F_A) \subseteq c(F_A) \), for every \( F_A \in \text{FSS}(X,E) \);
3. \((c)\) \( c(F_A \cup G_B) = c(F_A) \cup c(G_B) \), for every \( F_A, G_B \in \text{FSS}(X,E) \);
4. \((c)\) \( c(c(F_A)) = c(F_A) \), for every \( F_A \in \text{FSS}(X,E) \).

Then we can associate a fuzzy soft topology in the following way:

\( \tau = \{ F_A : F_A \in \text{FSS}(X,E) : c(F_A) = F_A \} \).

Moreover with this fuzzy soft topology \( \tau \), \( \bar{F_A} = c(F_A) \) for every \( F_A \in \text{FSS}(X,E) \).

Definition 2.16 [9]
Let \((X,E,\tau)\) be a fuzzy soft topological space and \( F_A \) be a fuzzy soft set over \((X,E)\). The fuzzy soft interior of \( F_A \), denoted by \( F_A^\circ \), is defined as the union of all fuzzy soft sets contained in \( F_A \). That is, \( F_A^\circ = \bigcup \{ G_B : G_B \text{ is fuzzy soft open set and } G_B \subseteq F_A \} \). Clearly, \( F_A^\circ \) is the largest fuzzy soft open set over \((X,E)\) contained in \( F_A \).

Theorm 2.17 [10]
Let \( i : \text{FSS}(X,E) \to \text{FSS}(X,E) \) be a function satisfying the following:

1. \((i)\) \( i(X) = X \);  
2. \((i)\) \( i(F_A) \subseteq F_A \), for every \( F_A \in \text{FSS}(X,E) \);  
3. \((i)\) \( i(F_A \cap G_B) = i(F_A) \cap i(G_B) \), for every \( F_A, G_B \in \text{FSS}(X,E) \);  
4. \((i)\) \( i(i(F_A)) = i(F_A) \), for every \( F_A \in \text{FSS}(X,E) \).

Then we can associate a fuzzy soft topology in the following way:

\( \tau = \{ F_A : F_A \in \text{FSS}(X,E) : i(F_A) = F_A \} \).

Moreover with this fuzzy soft topology \( \tau \), \( F_A^\circ = i(F_A) \) for every \( F_A \in \text{FSS}(X,E) \).

Definition 2.18 [3]
A pretopology on \( X \) can be described by a function \( a : 2^X \to 2^X \) which satisfies:

1. \((a)\) \( a(\phi) = \phi \);  
2. \((a)\) \( a(A) \supseteq A \), for every \( A \in 2^X \).

The pair \((X,a)\) is called a pretopological space.

Now, we define the concept of a soft pretopology as follows:

Definition 2.19
A soft pretopology on \((X,E)\) is a function \( a : \text{SS}(X,E) \to \text{SS}(X,E) \) which satisfies the following conditions:

1. \((a)\) \( a(\phi) = \phi \);  
2. \((a)\) \( a(F_A) \supseteq F_A \), for every \( F_A \in \text{SS}(X,E) \).

The triple \((X,E,a)\) is called a soft pretopological space.
Definition 2.20 [2]

A fuzzy pretopology on $X$ can be described by a function $a : I^X \to I^X$ which satisfies:

(PT 1) $a(\emptyset) = \emptyset$;

(PT 2) $a(A) \supseteq A$, for every $A \in I^X$.

The pair $(X,a)$ is called a fuzzy pretopological space.

Let $(X,a)$ be a fuzzy pretopological space, and let us consider the following properties:

(PT 3) $(X,a)$ is said to be of type $I$ if for every $A, B \in I^X$ such that $A \subseteq B$, we have $a(A) \subseteq a(B)$.

(PT 4) $(X,a)$ is said to be of type $D$ if for every $A, B \in I^X$ we have $a(A \cup B) = a(A) \cup a(B)$.

(PT 5) $(X,a)$ is said to be of type $S$ if for every $A \in I^X$ we have $a^2(A) = a(a(A)) = a(A)$.

One may notice that $(PT 4) \Rightarrow (PT 3)$, and any fuzzy pretopological space of type $D,S$ is a fuzzy topological space where $a$ is Kuratowski closure.

Definition 2.21 [2]

Let $(X,a)$ be a fuzzy pretopological space, the fuzzy interior function $i_a : I^X \to I^X$ is defined by $i_a(A) = (a(A))^\circ$.

The fuzzy interior function $i_a$ satisfies the following properties:

(1) $i_a(\emptyset) = \emptyset$;

(2) For every $A \in I^X$, $i_a(A) \subseteq A$;

(3) For every $A, B \in I^X$ if $A \subseteq B$, then $i_a(A) \subseteq i_a(B)$;

(4) For every $A, B \in I^X$, $i_a(A \cap B) = i_a(A) \cap i_a(B)$;

(5) For every $A \in I^X$, $i_a^2(A) = i_a(A)$.

Definition 2.22 [2]

Let $(X,a)$ be a fuzzy pretopological space and $A$ be a fuzzy subset of $X$. The trace of $a$ on $A$, denoted by $a_A$, is defined by $a_A(B) = a(B) \cap A$, for every fuzzy subset $B$ of $A$.

3. FUZZY SOFT PRETOPOLOGY

In this section we define fuzzy soft pretopology, fuzzy soft interior function, fuzzy soft pre-open set, fuzzy soft pre-closed set, the trace of a fuzzy soft pretopology and we study some of its properties.

Definition 3.1

A fuzzy soft pretopology on $(X,E)$ is a function $a : FSS(X,E) \to FSS(X,E)$ which satisfies the following conditions:

(PT 1) $a(\emptyset) = \emptyset$;

(PT 2) $a(F_A) \supseteq F_A$, for every $F_A \in FSS(X,E)$.

The triple $(X,E,a)$ is then said to be a fuzzy soft pretopological space.

A fuzzy soft pretopological space $(X,E,a)$ is said to be of:

(PT 3) Type $I$ : if for every $F_A, G_B \in FSS(X,E)$ such that $F_A \supseteq G_B$, we have $a(F_A) \supseteq a(G_B)$;

(PT 4) Type $D$ : if for every $F_A, G_B \in FSS(X,E)$, we have $a(F_A \cup G_B) = a(F_A) \cup a(G_B)$;

(PT 5) Type $S$ : if for every $F_A \in FSS(X,E)$, we have $a^2(F_A) = a(a(F_A)) = a(F_A)$.

We say that $a$ is of type $\sigma \eta$ if $\sigma$ is of type $\sigma$ and $\eta$, where $\sigma, \eta \in \{I,D,S\}$.
One notice that $(PT\ 4)\Rightarrow (PT\ 3)$ and any fuzzy soft pretopological space $(X, E, a)$ of type $DS$ is a fuzzy soft topological space where $a$ is Kuratowski closure.

**Proposition 3.2**

Let $a, b$ be two fuzzy soft pretopologies on $(X, E)$. Then

(1) The composite function of $a, b$ denoted by $a\circ b$ is a fuzzy soft pretopology on $(X, E)$.

(2) If $a, b$ are of type $I$ (resp. type $D$), then the composite function $a\circ b$ of $a$ and $b$ is of type $I$ (resp. type $D$).

**Proof:**

(1) $(a\circ b)(\tilde{\phi}) = a(b(\tilde{\phi})) = a(\tilde{\phi}) = \tilde{\phi}$. Also, $(a\circ b)(F_A) = a(b(F_A))$. Put $b(F_A) \subseteq G_B$, then $(a\circ b)(F_A) = a(G_A) \supseteq G_A = b(F_A) \supseteq F_A$. Therefore $(a\circ b)(F_A) \supseteq F_A$.

(2) (i) Let $a, b$ of type $I$ and $F_A \subseteq G_B$, then $b(F_A) \subseteq b(G_B)$ and $a(b(F_A)) \subseteq a(b(G_B))$. Therefore $(a\circ b)(F_A) = a(b(F_A)) \subseteq a(b(G_B))$.

This shows that $a\circ b$ is of type $I$.

(ii) Let $a, b$ be of type $D$, then $a(F_A \cup G_B) = a(F_A) \cup a(G_B)$ and $b(F_A \cup G_B) = b(F_A) \cup b(G_B)$. Now, $(a\circ b)(F_A \cup G_B) = a(b(F_A) \cup b(G_B)) = a(b(F_A)) \cup a(b(G_B)) = (a\circ b)(F_A) \cup (a\circ b)(G_B)$.

Therefore, $a\circ b$ is of type $D$.

**Definition 3.3**

(1) Let $a_1, a_2$ be two fuzzy soft pretopologies on $(X, E)$. $a_1$ is said to be coarser than $a_2$, written $a_1 \preceq a_2$, if $a_1(F_A) \subseteq a_2(F_A)$, for every $F_A \in FSS(X, E)$.

(2) Let $\{a_t : t \in T\}$ be a family of fuzzy soft pretopologies on $(X, E)$, where $T$ is an indexed set. We define $\sup a_t = \vee_{i \in T} a_t$, the least upper bound fuzzy soft pretopology on $(X, E)$ and $\inf a_t = \wedge_{i \in T} a_t$, the greatest lower bound fuzzy soft pretopology on $(X, E)$, as follows:

\[
(\sup_{i \in T} a_t)(F_A) = \sup_{i \in T} a_t(F_A) \text{ and } (\inf_{i \in T} a_t)(F_A) = \inf_{i \in T} a_t(F_A).
\]

**Proposition 3.4**

Let $\{a_t : t \in T\}$ be a family of fuzzy soft pretopologies on $(X, E)$. Then,

(1) If $a_t$ is of type $I$, for every $t \in T$, then $\sup_{i \in T} a_t$ and $\inf_{i \in T} a_t$ are also of type $I$.

(2) If $a_t$ is of type $D$, for every $t \in T$, then $\sup_{i \in T} a_t$ is also of type $D$.

(3) If $a_t$ is of type $S$, for every $t \in T$, then $\inf_{i \in T} a_t$ is also of type $S$.

**Proof:**

Let $F_A, G_B \in FSS(X, E)$.

(1) Let $a_t$ be of type $I$ and $F_A \subseteq G_B$, then $a_t(F_A) \subseteq a_t(G_B)$.

\[
(\sup_{i \in T} a_t)(F_A) = \sup_{i \in T} a_t(F_A) \subseteq \sup_{i \in T} a_t(G_B) = (\sup_{i \in T} a_t)(G_B).
\]

This implies that $\sup_{i \in T} a_t(F_A) \subseteq (\sup_{i \in T} a_t)(G_B)$ and therefore $\sup_{i \in T} a_t$ is of type $I$.

(ii) $(\inf_{i \in T} a_t)(F_A) = \inf_{i \in T} a_t(F_A) \subseteq \inf_{i \in T} a_t(G_B) = (\inf_{i \in T} a_t)(G_B)$. This implies that $(\inf_{i \in T} a_t)(F_A) \subseteq (\inf_{i \in T} a_t)(G_B)$ and therefore $\inf_{i \in T} a_t$ is of type $I$.

(2) Let $a_t$ be of type $D$, then $a_t(F_A \cup G_B) = a_t(F_A) \cup a_t(G_B)$. Now,
(\checkmark \bigcup_{i \in \mathbb{F}} a_i(F_A \cup G_B) = \bigcup_{i \in \mathbb{F}} a_i(F_A \cup G_B) \\
= \bigcup_{i \in \mathbb{F}} a_i(F_A) \cup \bigcup_{i \in \mathbb{F}} a_i(G_B) \\
= (\bigcup_{i \in \mathbb{F}} a_i(F_A)) \cup (\bigcup_{i \in \mathbb{F}} a_i(G_B)) \\
= ((\bigcup_{i \in \mathbb{F}} a_i(F_A)) \cup (\bigcup_{i \in \mathbb{F}} a_i(G_B))) \\
This shows that \checkmark a_i is of type D.

(3) Let a_i be of type S, then \checkmark a_i^2(F_A) = a_i(F_A)^2. Now,

(\checkmark a_i^2)(F_A) = (\checkmark a_i)(\checkmark a_i(F_A)) \\
= \checkmark a_i(\checkmark a_i(F_A)) \\
= (\checkmark a_i^2(F_A)) \cap (\checkmark a_i^2(F_A)) \\
= (\checkmark a_i^2(F_A)) \cap (\checkmark a_i a_i(F_A)) \\
= (\checkmark a_i a_i(F_A)) = (\checkmark a_i)(F_A).

Therefore (\checkmark a_i^2)(F_A) \subseteq (\checkmark a_i)(F_A).

On the other hand since \checkmark a_i(F_A) \supseteq F_A , we have (\checkmark a_i)(F_A) \supseteq F_A , which implies that (\checkmark a_i^2)(F_A) \supseteq (\checkmark a_i)(F_A).

Therefore, we have (\checkmark a_i^2)(F_A) = (\checkmark a_i)(F_A) and thus, \checkmark a_i is of type S.

**Definition 3.5**

Let \alpha be a fuzzy soft pretopology, we define the fuzzy soft interior function i_{\alpha}: FSS(X,E) \to FSS(X,E) by:

i_{\alpha}(F_A) = (\alpha(F_A)^C)^C.

The function i_{\alpha} satisfies the following properties:

(PT 1) i_{\alpha}(\emptyset) = \emptyset;

(PT 2) For every F_A \in FSS(X,E), we have i_{\alpha}(F_A) \subseteq F_A;

(PT 3) For every F_A,G_B \in FSS(X,E) such that F_A \subseteq G_B, we have i_{\alpha}(F_A) \subseteq i_{\alpha}(G_B);

(PT 4) For every F_A,G_B \in FSS(X,E), we have i_{\alpha}(F_A \cap G_B) = i_{\alpha}(F_A) \cap i_{\alpha}(G_B);

(PT 5) For every F_A \in FSS(X,E), we have i_{\alpha}^2(F_A) = i_{\alpha}(F_A).

**Definition 3.6**

Let \alpha be a fuzzy soft pretopology and F_A \in FSS(X,E). Then F_A is said to be a fuzzy soft pre-open (resp. fuzzy soft pre-closed) set in (X,E,\alpha) if i_{\alpha}(F_A) = F_A (resp. a_{\alpha}(F_A) = F_A).

It is clear that F_A is a fuzzy soft pre-closed set if and only if F_A^c is fuzzy soft pre-open.

**Proposition 3.7**

Let a_1,a_2 be two fuzzy soft pretopologies on (X,E). If a_1 \supseteq a_2 and F_A is fuzzy soft pre-open (resp. fuzzy soft pre-closed) with respect to a_2, then F_A is fuzzy soft pre-open (resp. fuzzy soft pre-closed) with respect to a_1.

**Proof:**

First, let F_A be fuzzy soft pre-open with respect to a_2 and a_1 \supseteq a_2, then a_{\alpha}(F_A) \supseteq a_{\alpha}(F_A) which implies that i_{\alpha}(F_A) \supseteq i_{\alpha}(F_A). Since F_A is fuzzy soft pre-open with respect to a_2, then i_{\alpha}(F_A) = F_A and therefore i_{\alpha}(F_A) \supseteq F_A.

Since i_{\alpha}(F_A) \supseteq F_A, then i_{\alpha}(F_A) = F_A and so F_A is fuzzy soft pre-open with respect to a_1.
Now, Let $F_A$ be fuzzy soft pre−closed with respect to $a_2$ and $a_1 \leq a_2$, then $a_1(F_A) \subseteq a_2(F_A)$. Since $F_A$ is fuzzy soft pre−closed with respect to $a_1$, then $a_1(F_A) = F_A$ and therefore $a_2(F_A) \subseteq F_A$. Since $a_1(F_A) \supseteq F_A$, then $a_1(F_A) = F_A$ and so, $F_A$ is fuzzy soft pre−closed with respect to $a_1$.

**Proposition 3.8**

Let $a$ be a fuzzy soft pretopology of type $D$, then the finite union of fuzzy soft pre−closed sets in $(X,E,a)$ is fuzzy soft pre−closed.

**Proof:**

Let $a$ be of type $D$ and $F_A,G_B$ fuzzy soft pre−closed sets in $(X,E,a)$, then $a(F_A) = F_A$ and $a(G_B) = G_B$. Also $a(F_A \cup G_B) = a(F_A) \cup a(G_B) = F_A \cup G_B$. This shows that $F_A \cup G_B$ is fuzzy soft pre−closed. Therefore, the finite union of fuzzy soft pre−closed sets in $(X,E,a)$ is fuzzy soft pre−closed.

**Definition 3.9**

Let $a$ be a fuzzy soft pretopology and $F_A \in FSS(X,E)$. The trace of $a$ on $F_A$, denoted by $a_{F_A}$, is defined as follows:

$$a_{F_A}(G_B) = a(G_B) \cap F_A,$$

for every fuzzy soft subset $G_B$ of $F_A$.

**Proposition 3.10**

The trace $a_{F_A}$ defines a fuzzy soft pretopology. Moreover if $a$ is of type $I$ (resp. $D,I,S$) then $a_{F_A}$ is of type $I$ (resp. $D,I,S$).

**Proof:**

First, we prove that $a_{F_A}$ is a fuzzy soft pretopology

1. $a_{F_A}(\emptyset) = a(\emptyset) \cap F_A = \emptyset \cap F_A = \emptyset$.

2. $a_{F_A}(G_B) = a(G_B) \cap F_A \supseteq a(G_B) \cap a(F_A) = G_B$. Therefore $a_{F_A}(G_B) \supseteq G_B$.

From (1) and (2), $a_{F_A}$ is a fuzzy soft pretopology.

(i) Suppose that $a$ is of type $I$ and $G_B, H_C \subseteq F_A$ such that $G_B \subseteq H_C$. Then $a(G_B) \subseteq a(H_C)$. Now, $a_{F_A}(G_B) = a(G_B) \cap F_A \subseteq a(H_C) \cap F_A = a_{F_A}(H_C)$. This shows that $a_{F_A}(G_B) \subseteq a_{F_A}(H_C)$. Since $G_B \subseteq H_C$ implies $a_{F_A}(G_B) \subseteq a_{F_A}(H_C)$, then $a_{F_A}$ of type $I$.

(ii) Suppose that $a$ is of type $D$, then $a(G_B \cup H_C) = a(G_B) \cup a(H_C)$. Now, $a_{F_A}(G_B \cup H_C) = a(G_B \cup H_C) \cap F_A = (a(G_B) \cup a(H_C)) \cap F_A$

$$= (a(G_B) \cap F_A) \cup (a(H_C) \cap F_A)$$

$$= a_{F_A}(G_B) \cup a_{F_A}(H_C)$$

Therefore $a_{F_A}$ of type $D$.

(iii) Suppose that $a$ is of type $I,S$, then we have (with $G_B \subseteq F_A$)

$$a^2_{F_A}(G_B) = a_{F_A}(a_{F_A}(G_B)) = a(a(G_B) \cap F_A) \cap F_A$$

which implies that $a(a(G_B) \cap F_A) \subseteq a^2(G_B) = a(G_B)$. So $a^2_{F_A}(G_B) = a(a(G_B) \cap F_A) \cap F_A \subseteq a(G_B) \cap F_A = a_{F_A}(G_B)$.

But $a_{F_A}(a_{F_A}(G_B)) \supseteq a_{F_A}(G_B)$, and these two inclusions prove that $a^2_{F_A}(G_B) = a_{F_A}(G_B)$, so $a_{F_A}$ of type $I,S$.
4. FUZZY SOFT PRETOPOLOGIES GENERATED BY FUZZY SOFT PRENEIGHBOURHOODS

In this section, we introduce and study the concepts of the fuzzy soft preneighbourhood system at a soft point, the degree of soft non-vacuity, soft \( \alpha \)-cut and generate fuzzy soft pretopologies by fuzzy soft preneighbourhoods.

Definition 4.1

The fuzzy soft preneighbourhood system at a soft point \( x_e \) in \( (X,E) \), denoted by \( \beta(x_e) \), is the family of all fuzzy soft subsets \( V_M \) which satisfy \( \mu_{\beta}(x_e) = 1 \).

Definition 4.2

A function \( \Psi : FSS(X,E) \rightarrow I \) is said to be a degree of soft non-vacuity \( ( \text{it associates to every fuzzy soft subset a number which represents the fact that it is more or less empty fuzzy soft set}) \) if it satisfies:

1. \( \Psi(\phi) = 0 \)
2. \( \Psi(G_B) = \sup_{x_e \in (X,E)} \mu_{\beta}(x_e) \)
3. \( G_B \subseteq H_C \Rightarrow \Psi(G_B) \leq \Psi(H_C) \)

In particular \( \Psi(G_B) = 1 \) if there exists a soft point \( x_e \) such that \( \mu_{\beta}(x_e) = 1 \).

In order to build the adherence we take a similar way as for the classical case, but to qualify how the intersection \( V_M \cap G_B \), with \( V_M \in \beta(x_e) \), is more or less empty we use the degree of soft non-vacuity.

Proposition 4.3

The function \( \alpha : FSS(X,E) \rightarrow FSS(X,E) \) build by \( \mu_{\alpha(G_B)} = \inf_{V_M \in \beta(x_e)} \psi(V_M \cap G_B) \) is a fuzzy soft adherence (fuzzy soft pretopology) of type \( I \).

Proof:

First note that \( \mu_{\alpha(G)}(x_e) = \inf_{V_M \in \beta(x_e)} \psi(V_M \cap G) = \inf_{V_M \in \beta(x_e)} \psi(\phi) = 0 \).

Then \( \mu_{\alpha(G)}(x_e) = 0 \) for every \( x_e \in (X,E) \). Thus \( \alpha(\phi) = \phi \).

Since for every \( x_e \in (X,E) \) and for every \( V_M \in \beta(x_e) \), we have \( \mu_{\alpha(G)}(x_e) = 1 \), then

\[
\mu_{\alpha(G)}(x_e) = \inf_{V_M \in \beta(x_e)} \psi(V_M \cap G_B) \geq \inf_{V_M \in \beta(x_e)} \psi(G_B) = \psi(G_B) = \sup_{x_e \in (X,E)} \mu_{\beta}(x_e) \geq \mu_{\beta}(x_e)
\]

This shows that \( \mu_{\alpha(G)}(x_e) \geq \mu_{\beta}(x_e) \) and therefore \( \alpha(G_B) \leq G_B \).

Let \( G_B, H_C \) be arbitrary fuzzy soft subsets with \( G_B \leq H_C \). Then for every \( V_M \in \beta(x_e) \), we have \( G_B \cap V_M \leq H_C \cap V_M \) which implies that \( \psi(G_B \cap V_M) \leq \psi(H_C \cap V_M) \).

Therefore \( \inf_{V_M \in \beta(x_e)} \psi(G_B \cap V_M) \leq \inf_{V_M \in \beta(x_e)} \psi(H_C \cap V_M) \), and so \( \mu_{\alpha(G)}(x_e) \leq \mu_{\alpha(H)}(x_e) \). Thus \( \alpha(G_B) \leq \alpha(H) \).

From the above, we conclude that \( \alpha \) is a fuzzy soft adherence (fuzzy soft pretopology) of type \( I \).

Definition 4.4

Let \( G_B \) be a fuzzy soft subset of \( (X,E) \), and let \( \alpha \in [0,1] \). The soft \( \alpha \)-cut of \( G_B \) is the soft subset \( (G_B)_\alpha \) of \( (X,E) \) defined by:

\[
(G_B)_\alpha = \{ x_e \in (X,E) : \mu_{\beta}(x_e) \geq \alpha \}.
\]

Now, we give another way to build a fuzzy soft adherence and we will later see that this two ways can give the same result.

Let us suppose that with every \( \gamma \in [0,1] \) is associated a classical type \( I \) soft pretopology \( (a)_\gamma \) and that if \( \gamma_1 \geq \gamma_2 \) we have \( (a)_{\gamma_1} \leq (a)_{\gamma_2} \).
Proposition 4.5

Let \((\alpha)_\gamma\) be a family of soft adherences of type \(I\) which satisfies \(\gamma_1 \geq \gamma_2\) implies \((\alpha)_{\gamma_1} \preceq (\alpha)_{\gamma_2}\). Then the function \(a': \text{FSS}(X,E) \rightarrow \text{FSS}(X,E)\) defined by the two equivalent ways:

\[(1) \mu_{x_\gamma}(x_e) = \sup \{\gamma : x_e \in (\alpha)_{\gamma}((G_{\beta})_{\alpha})\}\]

\[(2) (a'(G_{\beta}))_{\alpha} = \bigcap_{\gamma < \alpha} (\alpha)_{\gamma}((G_{\beta})_{\alpha})\]

is a type-\(I\) fuzzy soft pretopology.

Proof:

First we show that if \(\gamma_1 \geq \gamma_2\) then \((\alpha)_{\gamma_1}((G_{\beta})_{\alpha}) \subseteq (\alpha)_{\gamma_2}((G_{\beta})_{\alpha})\).

Since \((G_{\beta})_{\alpha} \preceq (G_{\alpha})_{\alpha}\) and \((\alpha)_\gamma\) of type \(I\), then \((\alpha)_{\gamma_1}((G_{\beta})_{\alpha}) \subseteq (\alpha)_{\gamma_2}((G_{\beta})_{\alpha})\).

Now, \((\alpha)_{\gamma_1} \preceq (\alpha)_{\gamma_2}\) implies \((\alpha)_{\gamma_1}((G_{\beta})_{\alpha}) \subseteq (\alpha)_{\gamma_2}((G_{\beta})_{\alpha})\). Therefore, we have \((\alpha)_{\gamma_1}((G_{\beta})_{\alpha}) \subseteq (\alpha)_{\gamma_2}((G_{\beta})_{\alpha})\).

But it is known that generally there is no membership function associated with \((\alpha)_{\gamma}((G_{\beta})_{\alpha})\) if \(\gamma \in [0,1]\) if we have not \((\alpha)_{\gamma}((G_{\beta})_{\alpha}) = (\alpha)_{\alpha}((G_{\beta})_{\alpha})\), and this for every \(\alpha\).

But we consider the fuzzy soft pretopology \(a'\) associated with \((a'(G_{\beta}))_{\alpha} = \bigcap_{\gamma < \alpha} (\alpha)_{\gamma}((G_{\beta})_{\alpha})\), which will satisfies \(a'(-\phi) = \phi\) and \(a'(G_{\beta}) \preceq G_{\beta}\) for every \(\gamma\) because \((a'(-\phi))_{\gamma} = \bigcap_{\gamma < \alpha} (\alpha)_{\gamma}((-\phi))_{\gamma}\) = \(\phi\), then \(a'(-\phi) = \phi\). Also, we have \((a'(G_{\beta}))_{\alpha} = \bigcap_{\gamma < \alpha} (\alpha)_{\gamma}((G_{\beta})_{\alpha})\), since \((\alpha)_{\gamma}((G_{\beta})_{\alpha}) \subseteq (G_{\beta})_{\alpha}\), then \((a'(G_{\beta}))_{\alpha} \preceq (G_{\beta})_{\alpha}\)\).

Thus \(a'(G_{\beta}) \preceq G_{\beta}\).

Now we prove that \(a'\) is of type \(I\), it is a consequence of the fact that the \((\alpha)_{\gamma}\) are of type \(I\).

We have \(G_{\beta} \preceq H_{\beta}\) implies \((G_{\beta})_{\alpha} \preceq (H_{\beta})_{\alpha}\), for every \(\gamma\).

Since \((\alpha)_{\gamma}\) of type \(I\), then \((\alpha)_{\gamma}((G_{\beta})_{\alpha}) \subseteq (\alpha)_{\gamma}((H_{\beta})_{\alpha})\), which implies that \((\alpha)_{\gamma}((G_{\beta})_{\alpha}) \subseteq (\alpha)_{\gamma}((H_{\beta})_{\alpha})\) and thus \(a'(G_{\beta}) \preceq a'(H_{\beta})\). This shows that \(a'\) is of type \(I\).

Now, we show that the two ways of building \(a'\) are equivalent.

We note that \(a''\) is the soft adherence built with \(\mu_{x_e}(x_e) = \sup \{\gamma : x_e \in (\alpha)_{\gamma}((G_{\beta})_{\alpha})\}\).

Let \(x_e\) be a soft point of \((a''(G_{\beta}))_{\alpha}\), then \(x_e \in (\alpha)_{\gamma}((G_{\beta})_{\alpha})\) for every \(\gamma < \alpha\).

(Because \(I = [0,1]\) is without hole), therefore \(x_e \in \bigcap_{\gamma < \alpha} (\alpha)_{\gamma}((G_{\beta})_{\alpha})\) and so \(x_e \in (a'(G_{\beta}))_{\alpha}\) \((1)\)

Let \(x_e \in (a'(G_{\beta}))_{\alpha}\), then \(x_e \in \bigcap_{\gamma < \alpha} (\alpha)_{\gamma}((G_{\beta})_{\alpha})\). Therefore \(x_e \in (\alpha)_{\gamma}((G_{\beta})_{\alpha})\) for every \(\gamma < \alpha\) and sup \(\{\gamma : x_e \in (\alpha)_{\gamma}((G_{\beta})_{\alpha})\} = \alpha\).

Thus \(x_e \in (a'(G_{\beta}))_{\alpha}\) \((2)\)

From \((1)\) and \((2)\) the two ways are equivalent.

Proposition 4.6

Let \(X\) be a universe set and \(E\) be a set of parameters, \(x_e\) be a soft point in \((X,E)\), \(\beta(x_e)\) be the family of all fuzzy soft preneighbourhoods of \(x_e\) and \((\beta(x_e))_{\gamma}\) its soft \(\gamma\)-cuts. So we can build a soft pretopology \((\alpha)_{\gamma}\) by \((\alpha)_{\gamma}((G_{\beta})_{\alpha}) = \{x_e : \forall (V_{\beta})_{\gamma} \in (\beta(x_e))_{\gamma}, (V_{\beta})_{\gamma} \subseteq (G_{\beta})_{\alpha} = \phi\} \) on \(\text{SS}(X,E)\).

We now consider the fuzzy soft adherences:

\[(1) a(G_{\beta}) \text{ defined by } \mu_{x_e}(x_e) = \inf_{\beta(x_e)} \psi(V_{\beta}) \cap G_{\beta}\]
(2) \( a'(G_B) \) defined by \( (a'(G_B) \gamma)\alpha = \tilde{\cap}_{\gamma \in \alpha} (a(G_B)\gamma) \), for every \( \alpha \in [0,1] \).

Then these two fuzzy soft adherences are identical.

**Proof:**

If for every \( \alpha \) we have \( x_\gamma \in (a'(G_B)\gamma) \), then \( (V_M \gamma)\alpha \cap (G_B)\gamma \) for every \( V_M \in \beta(x_\gamma) \). Therefore \( \psi(V_M \gamma) G_B \geq \alpha \) and so \( \liminf_{\gamma \in \alpha} \sup_{x \in (x,\gamma)} H_M \gamma (y_\gamma) = \tilde{\cap}_{\gamma \in \alpha} (a(G_B)\gamma) \).

Thus

\[
\limsup_{\gamma \in \alpha} \sup_{x \in (x,\gamma)} H_M \gamma (y_\gamma) = \delta_1 \geq \alpha .
\]

Let us consider \( \sup_{y \in \alpha} x_\gamma \in (a'(G_B)\gamma) \) for the same soft point \( x_\gamma \), we have \( \delta_1 \geq \delta_2 \).

Let us suppose that \( \delta_1 \geq \delta_2 \). This implies that there exist \( \alpha_1 \) such that \( \delta_1 \geq \alpha_1 > \delta_2 \), then \( \alpha_1 > \delta_2 \) and therefore \( x \neq (a(G_B)\gamma) \), then there exist \( V_M \in \beta(x_\gamma) \) such that \( (V_M \gamma)\alpha \cap (G_B)\gamma = \tilde{\phi} \). This implies that \( \psi(V_M \gamma) G_B \leq \alpha_1 \), and therefore \( \limsup_{\gamma \in \alpha} \sup_{x \in (x,\gamma)} H_M \gamma (y_\gamma) \leq \alpha_1 \), then \( \liminf_{\gamma \in \alpha} \inf_{x \in (x,\gamma)} H_M \gamma (y_\gamma) = \delta_1 \leq \alpha_1 \). This implies that \( \delta_1 \leq \alpha_1 \), which is a contradiction. Thus \( \delta_1 = \delta_2 \).

**Proposition 4.7**

Let \( X \) be a universe set, \( E \) be a set of parameters and for every \( \gamma \in \{0,1\} \), we associate a type-\( I \) soft adherence \( (a_\gamma) \) with the property: \( \gamma_1 \geq \gamma_2 \) implies \( (a_\gamma) \gamma_1 \leq (a_\gamma) \gamma_2 \).

We build the fuzzy soft type-\( I \) adherence \( a \) by: \( (a(G_B)\gamma)\alpha = \tilde{\cap}_{\gamma \in \alpha} (a(G_B)\gamma) \). If for every \( \gamma \), \( (a_\gamma) \) is of type \( D \) then \( a \) is of type \( D \). If for every \( \gamma \), \( (a_\gamma) \) is of type \( S \) then \( a \) is of type \( S \).

**Proof:**

We have shown that \((X,E,a)\) is of type-\( I \) fuzzy soft pretopology.

(1) Let us suppose that \( (a_\gamma) \) are of type \( D \), then

\[
(a(G_B \cap H_C)\gamma)\alpha = \tilde{\cap}_{\gamma \in \alpha} (a(G_B)\gamma \cap (H_C)\gamma) \]

\[
= \tilde{\cap}_{\gamma \in \alpha} (a(G_B)\gamma) \cap (\tilde{\cap}_{\gamma \in \alpha} (H_C)\gamma) \]

\[
= (a(G_B \cap H_C)\gamma)\alpha \cap (a(H_C)\gamma) \]

which prove that \( a \) is of type \( D \).

(2) Let \( (a_\gamma) \) of type \( S \).

Since \( a(G_B) \geq G_B \), then \( a^2(G_B) \geq a(G_B) \).

On the other hand since \( (a^2(G_B)\gamma)\mu \geq \tilde{\cap}_{\gamma \in \alpha} (a(G_B)\gamma) \) and since

\[
(a^2(G_B)\gamma)\mu \geq \tilde{\cap}_{\gamma \in \alpha} (a(G_B)\gamma) = \tilde{\cap}_{\gamma \in \alpha} (a(G_B)\gamma) \]

\[
= (a(G_B)\gamma)\mu \geq a(G_B) \]

which implies that \( a^2(G_B) \geq (a(G_B))^\geq \). So \( a^2(G_B) \geq a(G_B) \). Therefore, we have \( a^2(G_B) = a(G_B) \) and thus \( a \) is of type \( S \).

**REFERENCES**


