The Stability of Solution of Cauchy Problem for Evolution System in Euclidean Space $R^l, l > 2$

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Abstract
In this paper we discuss the existence and stability of solution of Cauchy problem for evolution system, in case when are changing a function that sets the initial value, and functional coefficients of equation. Also, we proved the existence of generalized (weak) solutions of quasi-linear evolution differential equations in spaces and its Hölder continuity with some additional assumption on the coefficients of these equations.

Indexing terms/Keywords

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TYPE (METHOD/APPROACH)
method of differential form;

INTRODUCTION
We study quasi-linear evolution system in divergence form in whole Euclidean space $R^l, l > 2$:

$$\frac{\partial}{\partial t} u^i + \lambda u^i - \frac{\partial}{\partial x_i} \left( a_{ij} (t, x, \bar{u}) \frac{\partial}{\partial x_j} u^i \right) + b^i (t, x, \bar{u}, \nabla \bar{u}) = f^i (t, x), \quad k = 1, \ldots, N$$

with initial condition

$$u(0, x) = \left( u^1, \ldots, u^N \right),$$

where is the unknown vector function $u^i (t, x) = \left( u^{i1}, \ldots, u^{iN} \right), \lambda > 0$ is real number and $f^i (t, x) = f^i = (f^1, \ldots, f^N)$ is given vector-function. $b^i (t, x, u, \nabla u) = b^i (t, x, \bar{u}, \nabla \bar{u})$ is vector-function of four variables: scalar, vector dimension $l$, vector dimension $N$, matrix dimension $l \times N$.

Measurable matrix $a_{ij} (t, x, u)$ dimension $l \times l$ satisfies ellipticity condition: $\exists \nu: 0 < \nu < \infty$ and executed the following inequality $\forall i, j \in \mathbb{N}, \exists \alpha, \beta > 0$:

$$\sum_{i=1}^{l} a_{ij} (t, x, u) \xi^i \xi_j \geq \alpha \sum_{i=1}^{l} \xi^2_i \geq \beta \sum_{i=1}^{l} a_{ij} (t, x, u) \xi^i \xi_j \quad \forall \xi \in R^l$$

We will call generalized (weak) solutions of quasi-linear differential system of parabolic type in $W^1_p (R^l, d^x)$ the element $u(t, x)$ that almost all $t \in [0, T], x \in R^l$, that

$$\nu \sum_{i=1}^{l} \xi^2_i \leq \sum_{i=1}^{l} a_{ij} (t, x, u) \xi^i \xi_j \quad \forall \xi \in R^l$$

We will call generalized (weak) solutions of quasi-linear differential system of parabolic type in $W^1_p (R^l, d^x)$ the element $u(t, x)$ that almost all $t \in [0, T]$ satisfies integral identity:

$$\left< u(t), v(t) \right>_0 = \int_0^t \left< u(t), v(t) \right> + \lambda \int_0^t \left< u(t), v(t) \right> + \int_0^t \left< f, v(t) \right> dt$$

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for almost all $t \in [0, T], x \in R^l$, that

$$\nu \sum_{i=1}^{l} \xi^2_i \leq \sum_{i=1}^{l} a_{ij} (t, x, u) \xi^i \xi_j \quad \forall \xi \in R^l$$

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$$\left< u(t), v(t) \right> + \lambda \int_0^t \left< u(t), v(t) \right> + \int_0^t \left< f, v(t) \right> dt = \int_0^t \left< f, \nabla u(t, x) \right> dt$$

for almost all $t \in [0, T], x \in R^l$, that

$$\nu \sum_{i=1}^{l} \xi^2_i \leq \sum_{i=1}^{l} a_{ij} (t, x, u) \xi^i \xi_j \quad \forall \xi \in R^l$$
and for any element \( v \in W^{1,0}_{1} \).

There are other concepts of solution, for example, its own solution of system (1) is a function \( u(t, x) \) if this function identically satisfies the system (1).

In general, studies on the stability of a system or solution of the parabolic equation can be considered several different options for stability. For example, let the system set parabolic form (1) with specified coefficients and conditions regarding these coefficients, also we have the initial condition at zero as \( u^{k}(0, x) = (u_{1}^{k},...,u_{N}^{k}) = u_{0} \) where \( u_{0} \) is known vector-function space variables, then we can consider, at least, three types of stability:

1. Stability, regarding the initial conditions is provided at the point where time-variable equal to zero, i.e., the change in the solution, vector-function \( u^{k}(t, x) = (u_{1}^{k},...,u_{N}^{k}) \) system (1), provided the initial condition \( u^{k}(0, x) = (u_{1}^{k},...,u_{N}^{k}) = u_{0} \) replaced by \( u^{k}(0, x) = (u_{1}^{k},...,u_{N}^{k}) = \varphi_{0} \) and \( \| \varphi_{0} - u_{0} \| \leq \varepsilon \) for given a positive \( \varepsilon \).

2. Stability, regarding the variation of coefficients, that small change of functions in the system (1), i.e., let given initial condition \( u^{k}(0, x) = (u_{1}^{k},...,u_{N}^{k}) = u_{0} \) then there is a solution \( u^{k}(t, x) = (u_{1}^{k},...,u_{N}^{k}) \) of given Cauchy problem, then say that in that same initial conditions \( u^{k}(0, x) = (u_{1}^{k},...,u_{N}^{k}) = u_{0} \) we considered other system

\[
\frac{\partial}{\partial t} u^{k} + \lambda^{k} u^{k} - \frac{\partial}{\partial x_{i}} \left( a_{j}^{k} (t,x,u^{k}) \frac{\partial}{\partial x_{j}} u^{k} \right) + b^{k}(t,x,u^{k},V u^{k}) = f^{k}(t,x), \quad k = 1,...,N \tag{6}
\]

provided that specified \( \lambda^{k}, a_{j}^{k}, b^{k}, f^{k} \) close, in a sense, to \( \lambda, a_{j}, b, f \).

3. The case which we will examine is a combination of the previous two, that is, the interpretation of stability of solution \( u(t, x) \) of Cauchy problem for the system (1), when we change a function that sets the initial value, and coefficients of equation.

2. The condition of coefficients equation (1)

Let matrix \( a_{ij}(t,x,u) \) is measurable dimension \( l \times l \) and satisfies ellipticity condition: \( \exists \nu: \quad 0 < \nu < \infty \) executed following inequality \( \nu \leq a(t,x,u) \), for almost all \( t \in [t,T], \quad x \in \mathbb{R}^{l} \).

We consider the conditions under which we study and parabolic system (1):

1. \( b(t,x,y,z) \) is measurable vector-function and its arguments \( b \in L^{1}_{loc}(\mathbb{R}) \);

2. vector-function \( b(t,x,y,z) \) satisfies almost everywhere, almost all \( t \in [0,T] \):

\[
|b(t,x,u,Vu)| \leq \mu_{l}(t,x)|Vu| + \mu_{2}(t,x)|u| + \mu_{3}(t,x). \tag{4}
\]

We introduce the class of functions for almost all \( t \in [0,T] \):

\[\text{PK}_{\beta}(A) = \{ f \in L^{1}_{loc}(\mathbb{R}^{l},d^{l}x) \mid |f(t,h)| \leq |\beta| V h + c(\beta)\|h\|_{l}^{2} \}, \quad \text{where} \quad \beta > 0, \quad c(\beta) \in \mathbb{R}^{l} \].

The condition (4) function \( \mu_{2}^{2} \in \text{PK}_{\beta}(A), \quad \mu_{2} \in \text{PK}_{\beta}(A) \), function \( \mu_{3} \in L^{1}(\mathbb{R}^{l}) \).

3. The growth vector-function \( b(t,x,y,z) \) almost everywhere satisfies the condition almost all \( t \in [0,T] \) (vector-function \( b(t,x,y,z) \) almost everywhere is continuous at \( y, z \):

\[
|b(t,x,u,Vu) - b(t,x,v,Vv)| \leq \mu_{l}(t,x)|Vu - Vv| + \mu_{2}(t,x)|u - v| \tag{5}
\]

where \( \mu_{2}^{2} \in \text{PK}_{\beta}(A), \quad \mu_{3} \in \text{PK}_{\beta}(A) \).

Remark. These types of conditions satisfied by functions Coulomb potential.
3. Research stability conditions of solution of evolution equation (1)

We study stability of solution $u$, in general case, that are changing function that sets the initial value, and coefficients of equation.

**Theorem 1.** Let given the sequence of the equations:

$$
\frac{\partial}{\partial t} u^i + \lambda^i u^i - \frac{\partial}{\partial x_j} \left( a^j_{ij} (t, x, u^i) \frac{\partial}{\partial x_j} u^i \right) + b^i (t, x, u, \nabla u) = f^i (t, x), \quad z = 1, 2, ..., k = 1, ..., N \tag{7}
$$

and each of which satisfies the conditions of existence of the solution of the system (1) with the same restrictions for all parameter values $z = 1, 2, ...,$. Let the sequence generalized solution $u^i \in V_{10}^2$, $z = 1, 2, ...,$ of Cauchy problems for systems (7) with the initial conditions $u^{i\circ} (0, x) = \left( u^{i\circ}_0, ..., u^{i\circ}_N \right) = \varphi_{0\circ}$. Let the following conditions:

$$
\lim_{\tau \to \infty} \left| u_{\tau} - \varphi_{0\circ} \right| = 0, \text{ i.e. sequence initial conditions limit to the original terms of the Cauchy problem for the system (1)};
$$

$$
\lim_{\tau \to \infty} \left| \sum_{j=1}^{N} a_{ij} - a_{ij\circ} \right| = 0; \lim_{\tau \to \infty} \left| \int_0^\tau f (\tau, \cdot, \nu, \nabla u) - b^i (\tau, \cdot, \nu, \nabla u) \right| \eta d\tau = 0
$$

these limits equality mean convergence coefficients (7) to the coefficients system (1), in fact in the last two terms convergence is weak (such conditions are considered in the first);

$$
|b^i (\tau, \cdot, \nu, \nabla u) - b^i (\tau, \cdot, u^i, \nabla u^i)| \leq \mu_i (\tau, \cdot) \left| V \left( u - u^i \right) \right| + \mu_i (\tau, \cdot) \left| u - u^i \right|
$$

condition form bounded function coefficients and $\mu_i (\tau, \cdot), \mu_i (\tau, \cdot)$ independent of $z$, that

$$
|b^i (\tau, \cdot, \nu, \nabla u) - b^i (\tau, \cdot, u^i, \nabla u^i)| \leq \mu_i (\tau, \cdot) \left| V \left( u - u^i \right) \right| + \mu_i (\tau, \cdot) \left| u - u^i \right|
$$

Then the sequence of generalized solutions $u^i \in V_{10}^2$, $z = 1, 2, ..., $ of Cauchy problems for systems (7) for the initial conditions $u^{i\circ} (0, x) = \left( u^{i\circ}_0, ..., u^{i\circ}_N \right) = \varphi_{0\circ}$ limit in $V_{10}^2$ to generalized solution $u(t, x)$ Cauchy problem for the system (1) with the initial conditions $u^i (0, x) = \left( u^i_0, ..., u^i_N \right) = u_0$.

**Proof.** The proof we will hold by the scheme: draw up integrated identity, according to the interpretation $u(t, x)$ of solution of Cauchy problem for the system (1) with initial conditions $u^i (0, x) = \left( u^i_0, ..., u^i_N \right) = u_0$ and for a sequence of generalized solutions $u^i \in V_{10}^2$, $z = 1, 2, ..., $ of Cauchy problems for systems (7) for the initial conditions $u^{i\circ} (0, x) = \left( u^{i\circ}_0, ..., u^{i\circ}_N \right) = \varphi_{0\circ}$, then subtract from an integrated identity for solution $u(t, x)$ integrated identity for $u^i \in V_{10}^2$ result written as an integrated identity for the difference $v^i = u - u^i$, then the proof is reduced to obtain estimates for $v^i = u - u^i$ and limit to $\lim_{\tau \to \infty} v^i = 0$ in the sense of $V_{10}^2$.

So we write integrated identity for system (1), at any element $\eta \in W_{10}^2$

$$
\left\langle u(t), \eta(t) \right\rangle_0^\tau + \int_0^\tau \left( - \left( \left( u(t), \frac{\partial}{\partial x_j} \eta(t) \right) \right) + \lambda^i \left( u(t), \eta(t) \right) \right) d\tau + \int_0^\tau \left( \sum_{j=1}^{N} a^j_{ij} \frac{\partial}{\partial x_j} u \right) d\tau + \int_0^\tau \left( \sum_{j=1}^{N} a^j_{ij} \frac{\partial}{\partial x_j} \eta \right) d\tau +
$$

$$
+ \int_0^\tau \left( f^i (t, \cdot, \eta) \right) d\tau = \int_0^\tau \left( f^i (t, \cdot, \eta) \right) d\tau.
$$

for any element $\eta \in W_{10}^2$ we obtain the following integral equality
\[ \{v^i(\tau), \eta(\tau)\} = \int_0^1 \left( -\{v^i(\tau), \partial_t \eta(\tau)\} + \lambda \{v^i(\tau), \eta(\tau)\} \right) d\tau + \int_0^1 \sum_{i,j=1}^{N} \left( a^i_{ij}(\tau, \tau) - a^i_{ij}(\tau, \tau) \right) \frac{\partial v^i}{\partial \mathbf{x}_j} u \frac{\partial \eta}{\partial \mathbf{x}_j} d\tau + \int_0^1 \sum_{i,j=1}^{N} a^j_{ij}(\tau, \tau) \frac{\partial v^i}{\partial \mathbf{x}_j} \frac{\partial \eta}{\partial \mathbf{x}_j} d\tau + \int_0^1 \left( b(\tau, \cdot, u, \nabla u) - b^i(\tau, \cdot, u^c, \nabla u), \eta \right) d\tau = \int_0^1 \left( f(\tau, \cdot) - f^i(\tau, \cdot), \eta \right) d\tau. \]

Evaluate each term separately, while using Hölder inequality.

We estimate term \( \int_0^1 \sum_{i,j=1}^{N} \left( a^i_{ij}(\tau, \tau) - a^i_{ij}(\tau, \tau) \right) \frac{\partial v^i}{\partial \mathbf{x}_j} u \frac{\partial \eta}{\partial \mathbf{x}_j} d\tau \) because

\[ \lim_{\tau \to \infty} \int_0^1 \sum_{i,j=1}^{N} \left( a^i_{ij}(\tau, \tau) - a^i_{ij}(\tau, \tau) \right) \frac{\partial v^i}{\partial \mathbf{x}_j} u \frac{\partial \eta}{\partial \mathbf{x}_j} d\tau = 0, \]

then

\[ \lim_{\tau \to \infty} \int_0^1 \sum_{i,j=1}^{N} \left( a^i_{ij}(\tau, \tau) - a^i_{ij}(\tau, \tau) \right) \frac{\partial v^i}{\partial \mathbf{x}_j} u \frac{\partial \eta}{\partial \mathbf{x}_j} d\tau = 0, \]

further we use the definition \( v^i = u - u^i \) and that \( v^i \in W^2_{1,0} \), we have

\[ \left\| \int_0^1 \sum_{i,j=1}^{N} \left( a^i_{ij}(\tau, \tau) - a^i_{ij}(\tau, \tau) \right) \frac{\partial v^i}{\partial \mathbf{x}_j} \frac{\partial \eta}{\partial \mathbf{x}_j} d\tau \right\| \leq \sum_{i,j=1}^{N} \left\| a^i_{ij}(\tau, \tau) \frac{\partial v^i}{\partial \mathbf{x}_j} \right\| \left\| \frac{\partial \eta}{\partial \mathbf{x}_j} \right\| . \]

First, we note that under the conditions of the theorem

\[ \lim_{\tau \to \infty} \int_0^1 \left( f(\tau, \cdot) - f^i(\tau, \cdot), \eta \right) d\tau = 0, \]

Next, the main task is to assess the nonlinear member using a form-bounded,

\[ \left| b(\tau, \cdot, u, \nabla u) - b^i(\tau, \cdot, u^c, \nabla u^c) \right| \leq \left| b(\tau, \cdot, u, \nabla u) - b^i(\tau, \cdot, u, \nabla u) \right| + \left| b^i(\tau, \cdot, u, \nabla u) - b^i(\tau, \cdot, u^c, \nabla u^c) \right| . \]

Because,

\[ \lim_{\tau \to \infty} \int_0^1 \left( b(\tau, \cdot, u, \nabla u) - b^i(\tau, \cdot, u, \nabla u), \eta \right) d\tau = 0, \]

that is, the first term tends to zero, then

\[ \left| b^i(\tau, \cdot, u, \nabla u) - b^i(\tau, \cdot, u^c, \nabla u^c) \right| \leq \mu_4(\tau, \cdot) \left\| \nabla v^i \right\| + \mu_5(\cdot, \cdot) \left\| v^i \right\| , \]

therefore, then we put \( \eta = v^i \) then, we estimate

\[ \left\| b^i(\tau, \cdot, u, \nabla u) - b^i(\tau, \cdot, u^c, \nabla u^c) \right\| \leq \| \mu_4 v^i \| \| \nabla v^i \| + \| \mu_5 v^i \| \| v^i \| \leq \frac{1}{2} \left( \frac{1}{\sigma^2} \| \mu_4 v^i \|^2 + \sigma^2 \| \nabla v^i \|^2 \right) + \frac{1}{2} \left( \frac{1}{\zeta^2} \| \mu_5 v^i \|^2 + \zeta^2 \| v^i \|^2 \right) \]

and we use forms - bounded coefficients

\[ \| \mu_4 v^i \| \leq \left( v^i \left( \mu_4 \right)^2 v^i \right) \leq \beta \| \nabla v^i \|^2 + c(\beta) \| v^i \|^2 \]

similarly estimated term of \( \mu_5 \). Then, we summarize similarly summands and choose the best constants and we prove the theorem. Theorem 2 is proved.
Remark. The condition of the theorem:  \( \lim_{\tau \to 0} \int_{\tau}^{\tau} \left( f(\tau, \cdot) - f^{\tau}(\tau, \cdot, \eta) \right) d\tau = 0 \); \( \lim_{\tau \to 0} \int_{\tau}^{\tau} \left( \int_{\tau}^{\tau} f(\tau, \cdot) - f^{\tau}(\tau, \cdot, \eta) \right) d\tau = 0 \) requiring only weak convergence of coefficients of the system, i.e., the weakest possible conditions.

4. The existence of the solution of the system (1)

We consider the following integral identity

\[
\langle u(\tau), v(\tau) \rangle_{\beta} + \int_0^\tau \left( \langle u(\tau), \partial_\tau v(\tau) \rangle + 2 \langle u(\tau), v(\tau) \rangle \right) d\tau + \int_0^\tau \left( \sum_{\gamma \in \mathbb{N}} a_{\gamma} \frac{\partial}{\partial \gamma} u_{\gamma}, \frac{\partial}{\partial \gamma} v_{\gamma} \right) d\tau + \int_0^\tau (h, v) d\tau = \int_0^\tau (f, v) d\tau
\]

for all \( u(\tau, x) \in W^\prime_{1, \beta}, \tau \in [0, T] \) and for any function \( v \in W^\prime_{1, \beta} \).

We rewrite identity (6) for \( \tau \in [0, T] \) in the form of

\[
\langle u(\tau), v(\tau) \rangle_{\beta} + \int_0^\tau \left( \langle u(\tau), \partial_\tau v(\tau) \rangle + \left( \sum_{\gamma \in \mathbb{N}} a_{\gamma} \frac{\partial}{\partial \gamma} u_{\gamma}, \frac{\partial}{\partial \gamma} v_{\gamma} \right) \right) d\tau = \int_0^\tau (f, v) d\tau - \int_0^\tau (\lambda \langle u(\tau), v(\tau) \rangle) d\tau - \int_0^\tau (h, v) d\tau.
\]

Let element \( v(\tau) = u|\tau|^{-2} \) and we estimate

\[
\langle u(\tau), u|\tau|^{-2} \rangle_{\beta} + \lambda \int_0^\tau \langle u(\tau), u|\tau|^{-2} \rangle d\tau + \int_0^\tau \left( \langle u(\tau), \partial_\tau u|\tau|^{-2} \rangle + \left( \sum_{\gamma \in \mathbb{N}} a_{\gamma} \frac{\partial}{\partial \gamma} u_{\gamma}, \frac{\partial}{\partial \gamma} u|\tau|^{-2} \right) \right) d\tau \leq
\]

\[
\int_0^\tau (f, u|\tau|^{-2}) d\tau + \int_0^\tau (\mu (t), |\nabla u| + \mu_2(t,x)|u| + \mu_3(t,x), u|\tau|^{-2}) d\tau.
\]

So, we have for the system (1) under (4) we obtained inequality (8). Further, we estimate each term on the right side separately

\[
\left| \langle f, u|\tau|^{-2} \rangle \right| \leq \|f\| \|u|\tau|^{-2} \| \leq \frac{\sigma^\tau}{p} \|u\| + \frac{1}{q \sigma^\tau} \|u|\tau|^{-2} \|
\]

\[
\left| \sum_{\gamma \in \mathbb{N}} a_{\gamma} \frac{\partial}{\partial \gamma} u_{\gamma}, \frac{\partial}{\partial \gamma} u|\tau|^{-2} \right| = \frac{4(p-1)}{p^2} \left( \nabla \left( u|\tau|^{-2} \right) \circ \nabla \left( u|\tau|^{-2} \right) \right).
\]

we use denoting of function \( w = u|\tau|^{-2} \) and respectively matrix \( \nabla w = \frac{P}{2} u|\tau|^{-2} \nabla u \).

\[
\mu_1 |\nabla u|, |u|^{-1} \leq \frac{2}{p} \langle \mu_1 |\nabla w|, |w| \rangle,
\]

\[
\mu_2(x), w^2 \leq \beta \langle \nabla w \circ \nabla w \rangle + c(\beta) \|w\|^2,
\]

\[
\mu_3(x), |u|^{-1} \leq \|\mu_3\| \|u|^{-1} \|
\]

we denote \( |u|^{-1} = \left( \sum_{\gamma \in \mathbb{N}} |u_{\gamma}|^p \right)^{\frac{1}{p}} \) and we obtain

\[
|u| |u|^{-1} = \left( \sum_{\gamma \in \mathbb{N}} |u_{\gamma}|^p \right)^{\frac{1}{p}} \left( \sum_{\gamma \in \mathbb{N}} |u_{\gamma}|^p \right)^{\frac{1}{p}} = \left( \sum_{\gamma \in \mathbb{N}} |u_{\gamma}|^p \right)^{\frac{1}{p}} = |u|^{-1}.
\]
Next use the Hölder and Young estimates \( \frac{2}{p} \langle \mu, |w| \rangle \leq \frac{2}{p} \mu \| |w| \|^2 \),

\[ \| \mu \| = \| (\mu, |w|) \|^2 \leq \left( \frac{\beta \langle \nabla w, \alpha \circ \nabla w \rangle + c(\beta) \| w \|^p}{p} \right)^\frac{1}{2}, \]

so

\[ \frac{2}{p} \langle \mu, |w| \rangle \leq \frac{2}{p} \mu \| |w| \|^2 \leq \frac{2}{p} \| \nabla w \|^2 \left( \frac{\beta \langle \nabla w, \alpha \circ \nabla w \rangle + c(\beta) \| w \|^p}{p} \right)^\frac{1}{2} \]

\[ \leq \frac{1}{p} \| \nabla w \|^2 + \epsilon^2 \left( \beta \langle \nabla w, \alpha \circ \nabla w \rangle + c(\beta) \| w \|^p \right) \]

\[ \leq \frac{1}{p} \left( \frac{1}{\epsilon^2} \langle \nabla w, \alpha \circ \nabla w \rangle + c(\beta) \| w \|^p \right) \]

Then we get estimations

\[ \int \left( (\partial_t u(t), (u|w|^{-2}(t))) + \sum_{\nu, \eta, \lambda} a_\nu \frac{\partial}{\partial \lambda} u, \frac{\partial}{\partial \lambda} (u|w|^{-2}(t)) \right) dt + \lambda \int \| u(t), u|w|^{-2}(t) \| dt \leq \]

\[ \leq \int \left( \frac{\sigma'}{p} \| f \|^2 + \frac{1}{q \sigma'} \| u|w|^{-2}(t) \| ^2 \right) dt \]

\[ + \int \left( \frac{1}{p} \| \nabla w \|^2 + \epsilon^2 \left( \beta \langle \nabla w, \alpha \circ \nabla w \rangle + c(\beta) \| w \|^p \right) \right) dt \]

\[ + \int \left( \beta \langle \nabla w, \alpha \circ \nabla w \rangle + c(\beta) \| w \|^p + \frac{1}{\gamma^2 q} \| w \|^2 + \frac{2}{p} \mu \| w \|^p \right) dt. \]

We use the equality \( \langle u(t), u|w|^{-2}(t) \rangle \) is a true for almost all real values \( t \)

\[ \frac{1}{p} \| f \|^2 + 4 \frac{p - 1}{p^2} \int \left( \langle \nabla w, \alpha \circ \nabla w \rangle \right) dt \leq \]

\[ \leq \int \left( \frac{\sigma'}{p} \| f \|^2 + \frac{1}{q \sigma'} \| u|w|^{-2}(t) \| ^2 \right) dt \]

\[ + \int \left( \frac{1}{p} \| \nabla w \|^2 + \epsilon^2 \left( \beta \langle \nabla w, \alpha \circ \nabla w \rangle + c(\beta) \| w \|^p \right) \right) dt \]

\[ + \int \left( \beta \langle \nabla w, \alpha \circ \nabla w \rangle + c(\beta) \| w \|^p + \frac{1}{\gamma^2 q} \| w \|^2 + \frac{2}{p} \mu \| w \|^p \right) dt. \]

Because of \( \| u|w|^{-2}(t) \| ^2 = \| u \| ^2 = \| w \|^2 \), then we obtain inequality...
Theorem 2. If the conditions (4), (5) is in space $W_1^2\left( (0,T) \times R^3 \right)$, then there is a solution of system (1).

Proof. We construct a sequence of approximate solutions $\{u^{\alpha}_{m}(t,x)\} = \{(u^1_{m},...,u^N_{m})\}$, $m = 1, 2,...,$ system

$$\frac{\partial}{\partial t} u^j + \lambda u^j - \frac{\partial}{\partial x_j} \left( a_{ij}(t,x,u) \frac{\partial}{\partial x_i} u^j \right) + b^j(t,x,u) = f^j, \quad k = 1, ..., N$$

which will be sought in the form $u^{\alpha}_{m}(t,x) = \{(u^1_{m},...,u^N_{m})\} = \{\sum_{i=1}^{N} c^m_i(\phi^{i}_{\alpha}(x)) \}$, where elements $\{\phi^{i}_{\alpha}(x)\}$ are determined by system of ordinary differential equations using substitution $u^{\alpha}_{m}(t,x) = \{(u^1_{m},...,u^N_{m})\} = \{\sum_{i=1}^{N} c^m_i(\phi^{i}_{\alpha}(x)) \}$
and initial conditions

\[ c_{mk}^n(0) = \left( a^k_n, \phi^k_n(x) \right), \quad n = 1, 2, \ldots, m. \]

The conditions of the problem implies that \[ |c_{mk}^n| \leq \text{const}, \quad n = 1, 2, \ldots, m \text{ for } t \in [0, T]. \] We will show that solutions uniformly bounded on \[ t \in [0, T] \] this follows from the limitations of the bottom elliptical matrix and conditions for nonlinear perturbation. We multiply \[ n \text{-th equation at } c_{mk}^n \text{ and on summands at } n \text{ from } 1 \text{ to } m, \] then we obtain inequality

\[
\frac{1}{2} \left\| u_n(t) \right\|^2 + \int_0^t \left\{ \nabla u_n \cdot a_n \nabla u_n \right\} d\tau + \lambda \int_0^t \left\| u_n \right\|^2 d\tau \leq \\
\leq \left( \frac{1}{\sqrt{\beta}} + c(\beta) \right) \int_0^t \left\| u_n \right\|^2 d\tau + \\
+ \sqrt{\beta} \left( 1 + \sqrt{\beta} \right) \int_0^t \left\{ \nabla u_n \cdot a_n \nabla u_n \right\} d\tau + \\
+ \sqrt{\beta} \frac{t}{2} \left\| f \right\|^2 d\tau + \sqrt{\beta} \frac{t}{2} \left\| \mu_2 \right\|^2 d\tau.
\]

Next, we use a known lemma.

**Lemma 1.** Let absolutely continuous at \[ t \in [0, T] \] positive function \[ \psi(t) \] such that \[ \psi(0) = 0 \] and almost all \[ t \in [0, T] \] satisfies the following inequality

\[
\frac{d}{dt} \psi(t) \leq c(t) \psi(t) + F(t)
\]

where functions \( c(t) \) and \( F(t) \) are positive and integrated at \( t \in [0, T] \). Then we have

\[
\psi(t) \leq \exp \left( \int_0^t c(\tau) d\tau \right) \left[ 
F(\tau) d\tau,
\right.
\]

and

\[
\frac{d}{dt} \psi(t) \leq c(t) \exp \left( \int_0^t c(\tau) d\tau \right) \left[ F(\tau) d\tau + F(t) \right].
\]

Thus, provided that \( u \in L^2(\mathbb{R}) \) is true we have a priori estimate

\[
\max_{t \in [0, T]} \sum_{n=1}^{\infty} \left( c_{mk}^n(t) \right)^2 = \max_{t \in [0, T]} \left\| u_n \right\|^2 \leq \text{const}.
\]

Functions \( c_{mk}^n(t) = c_{mk}^n(t) = (u^m_n(t, x), \phi^m_n(x)), \) \( m, n = 1, 2, \ldots \) is continuous on \( t \in [0, T] \). To study functions \( c_{mk}^n(t) = (u^m_n(t, x), \phi^m_n(x)), \) \( m, n = 1, 2, \ldots \) in the absolute continuity at \( t \in [0, T] \), we look at integrals at \( t \) to \( t + \Delta t \), we use estimation that was obtained above, so we have

\[
\langle \mu_n^m(t + \Delta t, x) - u_n^m(t, x), \phi^m_n \rangle \leq \\
\leq \int \left( \left\{ \sum_{i=1}^{\infty} a_{ni} \frac{\partial}{\partial x_i} u_n^m \phi^m_n \right\} \right) d\tau + \int \left\{ f, \phi^m_n \right\} d\tau + \lambda \int \left\{ u_n^m, \phi^m_n \right\} d\tau + \\
+ \int \left\{ \mu_1(t, x) \left| \nabla u_n^m \right| + \mu_2(t, x) u_n^m + \mu_3(t, x, \phi^m_n) \right\} d\tau \leq
\]
\[
\leq c_n \int_t^{t+\Delta t} \left( \sum_{i,j=1}^{\infty} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} u_n \right) dt - \lambda c_n \int_t^{t+\Delta t} \|u_n\| dt + 
+ c_n \text{const}(\beta) \int_t^{t+\Delta t} \|u_n\| dt + c_n \text{const}(\beta) \int_t^{t+\Delta t} (\nabla u_n \cdot a \cdot \nabla u_n) dt + 
+ c_n \text{const}(\beta) \left( \int_t^{t+\Delta t} \|u_n\| dt + \int_t^{t+\Delta t} \|u_n\| dt \right) \leq \text{const}(n, \varphi, l) \Delta t.
\]

Consequently, constants \( \text{Const}(n, \varphi, l) \) dependent on \( n, \varphi, l \) but do not depend on \( m \) provided \( m \geq n \) i.e. inequality

\[
\left| c_n^x (t + \Delta t) - c_n^x (t) \right| \leq \varepsilon (\Delta t) \|u_n\| \to 0.
\]

By diagonal way, we construct a subsequence \( c_{n(i)}^x, i = 1, 2, \ldots, \) coinciding evenly on \([0, T]\) to some continuous function \( c_n^x(t), n = 1, 2, \ldots, \) for everyone \( n \). The sequence of functions \( c_n^x(t), n = 1, 2, \ldots, \) determines the function \( u(t, x) \) by rule

\[
\phi(t, x) = \sum_{i=1}^{\infty} c_i^x (t) \varphi_i^x (x).
\]

The sequence of functions \( \{u^x_n(t, x)\} = \{u^x_1, u^x_2, \ldots, u^x_n\} = \left\{ \sum_{i=1}^{\infty} c_i^x (t) \varphi_i^x (x) \right\} \) limits to \( u^x(t, x) = \sum_{i=1}^{\infty} c_i^x (t) \varphi_i^x (x) \) weak in \( L^2(R') \) and evenly at \( t \in [0, T] \). We have

\[
(u^x_{m(i)} - u^x, v) = \sum_{i=1}^{\infty} (v^i, c_{m(i)}^x)(u^x_{m(i)} - u^x, \varphi_i^x) + 
+ \left( u^x_{m(i)} - u^x, \sum_{n=1}^{\infty} (v^i, c_n^x) \varphi_n^x \right),
\]

we use estimation

\[
\left| u^x_{m(i)} - u^x, \sum_{n=1}^{\infty} (v^i, c_n^x) \varphi_n^x \right| \leq \text{const} \left( \sum_{n=1}^{\infty} (v^i, c_n^x)^2 \right)^{\frac{1}{2}}.
\]

Let the number \( s \) sufficiently large, then for any positive number \( \varepsilon \) given in advance we have inequality

\[
\text{const} \left( \sum_{n=1}^{\infty} (v^i, c_n^x)^2 \right)^{\frac{1}{2}} \leq \varepsilon \frac{\varepsilon}{2},
\]

then for all \( t \in [0, T] \) for sufficiently large \( m(i) \) the first amount is also less \( \frac{\varepsilon}{2} \) to all \( t \in [0, T], \) that proved uniformly for all \( t \in [0, T], \) then sequence \( \{u^x_{m(i)}\} \) limits to \( u \) weakly in \( L^2(R') \) relatively \( t \in [0, T] \). There is a subsequence of the sequence \( \{u^x_{m(i)}\} \) which converges to \( u \) weak in \( L^2(R') \) with its derivatives \( \{\partial_j u_{m(i)}\}, \) again we denote it as \( \{u_n\}. \)

We show that the limit function \( u \) is the solution of the Cauchy problem for the system (1). We can write identity

\[
\langle u_n(t), v(t) \rangle \|_H + \int_0^t \langle -\langle u_n(t), \partial_j v(t) \rangle + \lambda \langle u_n(t), v(t) \rangle \rangle dt + 
+ \int_0^t \left( \sum_{i,j=1}^{\infty} a_{ij} \frac{\partial}{\partial x_i} u_n \frac{\partial}{\partial x_j} v \right) dt + \int_0^t (b, v) dt = \int_0^t (f, v) dt,
\]

which is true for any function \( v^i = \sum_{i=1}^{\infty} d^{m(i)}(t) \varphi_{m(i)}^i(x) \) where \( d^{m(i)}(t) \) - continuous functions of argument \( t \in [0, T], \) which have bounded generalized derivatives. The set of such functions is denoted \( \varphi_m \) and function \( u_n \) belongs to \( \varphi_m. \) Let set \( \varphi, \) is formed by the union at \( m \) sets \( \varphi_m \) is dense in \( W^1. \)
We have for any function \( \phi \),\( \int_0^1 \left( \langle u(t), \hat{\phi}(t) \rangle + \lambda \langle u(t), v(t) \rangle \right) dt + \int_0^1 \left( \sum_{i,j=1}^{N} a_{ij} \frac{\partial}{\partial x_j} u_m \frac{\partial}{\partial x_i} v \right) dt + \int_0^1 \langle (f, u(t)) \rangle dt = \int_0^1 \langle (f, v(t)) \rangle dt. \) (10)

It is true for any \( v \) element of the set \( \phi \).

Then we use the inequality monotony of type, that is, for any \( \phi \) element of the set \( \phi_m \), we must prove the following inequality

\[
\int_0^1 \sum_{i,j=1}^{N} a_{ij} \frac{\partial}{\partial x_j} u_m \frac{\partial}{\partial x_i} (u_m - \phi) \, dt + \text{function} (\|u_m - \phi\|) \geq 0.
\]

Indeed, let \( v = u_m - \phi \) then we have

\[
\int_0^1 \sum_{i,j=1}^{N} a_{ij} \frac{\partial}{\partial x_j} u_m \frac{\partial}{\partial x_i} (u_m - \phi) \, dt = \int_0^1 \langle (f, u_m - \phi) \rangle dt.
\]

and therefore

\[
\int_0^1 \sum_{i,j=1}^{N} a_{ij} \frac{\partial}{\partial x_j} u_m \frac{\partial}{\partial x_i} (u_m - \phi) \, dt = \int_0^1 \langle (f, u_m - \phi) \rangle dt.
\]

and then

\[
\int_0^1 \left( \langle u_m(t), \hat{\phi}(t) \rangle + \lambda \langle u_m(t), v(t) \rangle \right) dt + \int_0^1 \left( \sum_{i,j=1}^{N} a_{ij} \frac{\partial}{\partial x_j} u_m \frac{\partial}{\partial x_i} v \right) dt + \int_0^1 \langle (f, u_m(t)) \rangle dt - \int_0^1 \langle (f, v(t)) \rangle dt - \frac{1}{2} \|u_m\|_*^2 + \langle u_m, \phi \rangle u_m^* + \text{function} (\|u_m - \phi\|) \geq 0.
\]

The last inequality at a fixed function \( \phi \) almost all \( t \in [0, T] \) you can go to the limit \( m \to \infty \), then we get the following inequality
\[
\int_0^t \left( -\langle u(\tau), \partial_t (u - \varphi)(\tau) \rangle + \lambda \langle u(\tau), (u - \varphi)(\tau) \rangle \right) d\tau + \\
+ \int_0^t \left( \sum_{i,j=1}^{N} a_{ij} \frac{\partial}{\partial x_j} u \frac{\partial}{\partial x_i} (u - \varphi) \right) d\tau + \int_0^t \langle b, (u - \varphi) \rangle d\tau - \\
- \int_0^t \langle f, (u - \varphi) \rangle d\tau - \frac{1}{2} \| u \|_{H^4}^2 + (u, \varphi) \| u - \varphi \| + \text{function} (\| u - \varphi \|) \geq 0.
\] (11)

If in (10) to take \( v = u \) then we have (in order to make this replacement you must use estimates that were prepared earlier, since the function \( u \), generally speaking are not differentiated by \( t \in [0, T] \))

\[
\frac{1}{2} \| u \|_{H^4}^2 + \lambda \int_0^t \| u(\tau) \|^2 d\tau + \\
+ \int_0^t \left( \sum_{i,j=1}^{N} a_{ij} \frac{\partial}{\partial x_j} u \frac{\partial}{\partial x_i} (u - v) \right) d\tau + \int_0^t \langle b, (u - v) \rangle d\tau - \\
- \int_0^t \langle f, (u - v) \rangle d\tau + \text{function} (\| u - v \|) \geq 0.
\] (12)

We use (10) (11) (12) for arbitrary function \( v \in \varphi_{m} \) and any \( m \), so for any element \( v = \bigcup_{\omega = 1}^{\infty} \varphi_{m} \), we get

\[
\int_0^t \left( -\langle u(\tau), \partial_t (u - v)(\tau) \rangle + \lambda \langle u(\tau), (u - v)(\tau) \rangle \right) d\tau + \\
+ \int_0^t \left( \sum_{i,j=1}^{N} a_{ij} \frac{\partial}{\partial x_j} u \frac{\partial}{\partial x_i} (u - v) \right) d\tau + \int_0^t \langle b, (u - v) \rangle d\tau - \\
- \int_0^t \langle f, (u - v) \rangle d\tau + \text{function} (\| u - v \|) \geq 0.
\]

Since the set \( \varphi \) which is formed by the union \( m \) sets \( \varphi_{m} \) which is dense in \( W^2_1 \), then for any \( \varepsilon > 0 \) and for any function \( \varphi \in \varphi \) we can put \( v = u - \varepsilon \varphi \), then

\[
\varepsilon \int_0^t \left( -\langle u(\tau), \partial_t (u - \varepsilon \varphi)(\tau) \rangle + \lambda \langle u(\tau), (u - \varepsilon \varphi)(\tau) \rangle \right) d\tau + \\
+ \varepsilon \int_0^t \left( \sum_{i,j=1}^{N} a_{ij} \frac{\partial}{\partial x_j} u \frac{\partial}{\partial x_i} (u - \varepsilon \varphi) \right) d\tau + \varepsilon \int_0^t \langle b, \varphi \rangle d\tau - \\
- \varepsilon \int_0^t \langle f, \varphi \rangle d\tau + \text{function} (\varepsilon \| \varphi \|) \geq 0.
\]

We go to the limit \( \varepsilon \rightarrow 0 \), we obtain

\[
\int_0^t \left( -\langle u(\tau), \partial_t \varphi(\tau) \rangle + \lambda \langle u(\tau), \varphi(\tau) \rangle \right) d\tau + \\
+ \int_0^t \left( \sum_{i,j=1}^{N} a_{ij} \frac{\partial}{\partial x_j} u \frac{\partial}{\partial x_i} \varphi \right) d\tau + \int_0^t \langle b, \varphi \rangle d\tau - \int_0^t \langle f, \varphi \rangle d\tau \geq 0.
\]

As set \( \varphi \) is dense in \( W^2_1 \), then the last inequality implies that for every \( \varphi \in W^2_1 \) is true equality

\[
\int_0^t \left( -\langle u(\tau), \partial_t \varphi(\tau) \rangle + \lambda \langle u(\tau), \varphi(\tau) \rangle \right) d\tau + \\
+ \int_0^t \left( \sum_{i,j=1}^{N} a_{ij} \frac{\partial}{\partial x_j} u \frac{\partial}{\partial x_i} \varphi \right) d\tau + \int_0^t \langle b, \varphi \rangle d\tau - \int_0^t \langle f, \varphi \rangle d\tau = 0,
\]

which means that the element \( u \in W^2_1 \) is the solution of a given system (1).
Conclusions

We considered the existence of generalized solutions of quasi-linear evolution differential equations in $W^1_2([0,T] \times \mathbb{R}^1)$ space with conditions of form-bounded functional coefficients and we studied stability of this solution of Cauchy problem for evolution system, in case when are changing a function that sets the initial value, and functional coefficients of equation.

REFERENCES


