ON THE SOLUTIONS OF ILL-POSED CAUCHY PROBLEMS FOR SOME SINGULAR INTEGRO-PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. Singular integro-partial differential equations are studied, without any restrictions on the characteristic forms of the partial differential operators. A parabolic transform is defined and by using this transform, existence and stability results can be obtained. The Cauchy problem of fractional general partial differential equations can be considered as a special case from the obtained results. In addition, Hilfer fractional differential equations can be solved.

Mathematics Subjects Classification. 34 K 30,26 A 33, 60 H 15

Key words. Singular integro-partial differential equations, parabolic transform, Cauchy problem for general fractional differential equations, Hilfer partial fractional differential equations

Introduction
Consider the following singular integro-partial differential equations:

\[ \frac{\partial u(x,t)}{\partial t} = \psi(x,t) + \int_0^t L(x,t,\theta,D)u(x,\theta)d\theta, \]
\[ u(x,0) = \phi(x). \tag{1.1} \]

where

\[ L(x,t,\theta,D) = \sum_{|q|\leq m} a_q(x,t,\theta)D^q, \]
\[ D^q = D_1^{q_1} \ldots D_k^{q_k}, \quad D_j = \frac{\partial}{\partial x_j}, \quad j = 1, \ldots, k, \]
\[ q = (q_1, \ldots, q_k) \text{ is a multi-index, } |q| = q_1 + \cdots + q_k, \]
\[ x = (x_1, \ldots, x_k) \in \mathbb{R}^k, \quad \mathbb{R}^k \text{ is the } k \text{-dimensional Euclidean space.} \]

It is supposed that the coefficients \( a_q \) are continuous for all \( q, |q| \leq m \), on
\[ S = \{(x,t,\theta): x \in \mathbb{R}^k, \theta < t\}, \]
and:
\[ \int_0^t |a_q(x,t,\theta)|d\theta \leq M, \tag{1.3} \]
for some constant \( M > 0 \), \( M \) does not depend on \( x, t \) and \( \theta \).

If
\[ a_q(x,t,\theta) = \frac{1}{\Gamma(\alpha)} (t-\theta)^{\alpha-1} a^*_q, \]
where \( a^*_q \) are continuous bounded functions on \( \mathbb{R}^k \times [0,t] \), \( 0 < \alpha \leq 1 \), \( \Gamma(\alpha) \) is the gamma function,
then the integral part in equation (1.1) is transformed to a fractional integral of order \( \alpha \).

We can study as a special case of equation (1.1), the following Cauchy problem,

\[ \frac{\partial u(x,t)}{\partial t^\beta} = \sum_{|q|\leq m} a^*_q(x,t)D^q u(x,t), \]
\[ u(x,0) = \phi_1(x), \tag{1.4} \]
\[ \frac{\partial u(x,0)}{\partial t} = \phi_2(x), \tag{1.5} \]
\[ u(x,0) = \phi(x). \tag{1.6} \]

where \( 1 < \beta \leq 2, \phi_1 \) and \( \phi_2 \) are given continuous bounded functions on \( \mathbb{R}^k \).
It is well known that the Cauchy problem for elliptic equation is not correctly formulated. The results here are given without any restrictions on the characteristic form
\[ \sum_{|q|=m} a_q y^q, \]
where \( y = (y_1, ..., y_k) \). (see [12, 17, 26, 27]).

In section 2, we shall define the parabolic transform.

Using this transform, we shall solve equation (1.1) under suitable choices of the functions \( \phi \) and \( \psi \), (see [11]).

Stability results are also obtained.

In section 3, we shall solve equations of the form
\[ u(x, t) = \phi + \int_0^t L(x, t, \theta, D)u(x, \theta)d\theta, \]
where the coefficients \( a_q \)'s in (1.7) are continuous on \( S \) and satisfy the following inequality:
\[ |a_q(x, t, \theta)| \leq \frac{M}{(t-\theta)^{1-\alpha}}, \]
for some constant \( M > 0 \), and all \( q, |q| \leq m, (x, t, \theta) \in S, 0 < \alpha \leq 1 \).

Again the unique solution of the following fractional Cauchy problem:
\[ \frac{\partial^\alpha u(x, t)}{\partial t^\alpha} = \sum_{|q|=m} a_q^\alpha(x, t)D^\alpha u(x, t), \quad u(x, 0) = \phi(x), \]
can be obtained under suitable conditions on \( \phi \) and \( a_q^\alpha \) (0 < \( \alpha \leq 1 \)).

In section 4, we shall solve the following Hilfer fractional partial differential equations:
\[ D^\alpha_{\alpha, \nu} u(x, t) = \sum_{|q|=m} a_q^\alpha D^\alpha u(x, t), \]
with the initial conditions:
\[ I^\alpha(1-\nu)(1-\alpha) u(x, 0) = \phi(x), \]
where
\[ D^\nu_{\alpha, \nu} u(x, t) = \nu(1-\nu) \frac{d}{dt} I^\nu(1-\nu)(1-\alpha) u(x, t), \]
\[ I^\nu_\alpha u(x, t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{u(x, \tau)}{(t-\tau)^{\alpha-1}} d\tau. \]
The considered problems here are ill-posed.

There are many articles that contain extensive bibliographics as well as a survey of some important applications to ill-posed Cauchy problems arising in elastodynamics (see [23-25]).

2 Parabolic transform and existence theorems

Let \( C_c(R^k) \) be the set of all bounded continuous functions on \( R^k \).

Consider the following Cauchy problem:
\[ \frac{\partial u(x, t)}{\partial t} = [D^2_1 + \cdots + D^2_k]^{2N+1} u(x, t), \]
\[ u(x, 0) = \phi(x) \in C(R^k), \]
where \( N \) is a sufficiently large positive integer.

The solution of the Cauchy problem (2.1), (2.2) is given by
\[ u(x, t) = \int G(x - y, \tau) \phi(y) dy, \quad dy = dy_1 ... dy_k, \]
and the integral is taken over \( R^k \).

The function \( G \) is the fundamental solution of the Cauchy problem (2.1),(2.2).

For sufficiently large \( N \), we can find \( \gamma \in (0,1) \), and a constant \( M > 0 \) such that:
for all \( t > 0, |q| \leq M \).

A parabolic transform of a function \( F \) is a function \( \tilde{F} \) defined by

\[
\tilde{F}(x,t_1,...,t_r,c_1t+c_2) = \int G(x-y,c_1t+c_2)F(y,t_1,...,t_r)dy,
\]

where \( c_1 \geq 0, c_2 \geq 0, t_j, t \in [0,T], j = 1, ..., r \) and \( F(y,t_1,...,t_r) \) is a continuous bounded function on \( \mathbb{R}^r \times [0,T]^r \).

From (1.1), (1.2), we can write

\[
\int \ldots \sum_{|q| \leq M} b_q(x,t,\theta)d\theta = \int_0^t \ldots \sum_{|q| \leq M} b_q(x,t,\theta)d\theta,
\]

where

\[
b_q(x,t,\theta) = \int_0^t a_q(x,s,\theta)ds.
\]

We shall find a dense set \( E \) in \( C_b(\mathbb{R}^5) \) such that if \( \phi, \Psi \in E \), for every \( t \in [0,T] \), \( b_q \in E \), for every \( t, \theta \in [0,T] \), then equation (2.5) can be solved.

Also the solutions of (2.5) will continuously depend on \( \phi, \Psi \).

Let us study the following equation:

\[
\tilde{\vartheta}(x,t) = \tilde{\phi}(x,c_1t) + \int_0^t \tilde{\Psi}(x,\theta,c_1t)d\theta
\]

\[
+ \int_0^t \sum_{|q| \leq M} \tilde{b}_q(x,t,\theta,c_1t)D^q\tilde{\vartheta}(x,\theta,c_2-c_1\theta)d\theta,
\]

where \( c_1 \geq c_1T, c_1, c_2 \) are positive constants.

Theorem 2.1 If \( \phi \in C_b(\mathbb{R}^5), \Psi, b_q \in C_b(\mathbb{R}^5 \times [0,T]) \), then there exists a unique solution \( \vartheta \in C_b(\mathbb{R}^5 \times [0,T]) \) of equation (2.6).

**Proof.** Let us use the method of successive approximations.

Let \( \vartheta_n \) be a sequence of functions, defined by

\[
\vartheta_{n+1}(x,t) = \tilde{\phi}(x,c_1t) + \int_0^t \tilde{\Psi}(x,\theta,c_1t)d\theta
\]

\[
+ \int_0^t \sum_{|q| \leq M} \tilde{b}_q(x,t,\theta,c_1t)D^q\vartheta_n(x,\theta,c_2-c_1\theta)d\theta,
\]

where

\[
\vartheta_n(x,\theta,c_2-c_1\theta) = \int G(x-y,c_2-c_1\theta)\vartheta_n(y,\theta)dy.
\]

According to the semi-group property, one gets

\[
\vartheta_n(x,\theta,c_2-c_1\theta) = \int G(x-y,c_2-c_1\theta)\vartheta_n(y,\theta)dy,
\]

where

\[
\vartheta_n(x,\theta,c_2-c_1\theta) = \int G(x-y,c_2-c_1\theta)\vartheta_n(y,\theta)dy.
\]

Using (2.3), (2.7), (2.8), (2.9) and remembering that the coefficients are bounded, one can find a constant \( M > 0 \) such that

\[
\max_{x} \left| \vartheta_{n+1}(x,t) - \vartheta_n(x,t) \right| \leq M \int_0^t \max_{x} \left| \varphi(x,\theta) - \varphi(x,\theta) \right| \left| (x-\theta)^r \right| d\theta.
\]

Suppose that zero approximation \( \vartheta_0(x,t) \) is identically equal to zero.

Since \( \phi \) and \( \Psi \) are bounded functions, there exists a constant \( K > 0 \) such that
From (2.10) and (2.11), we get by induction:

$$\max_x|\vartheta_{n+1}(x,t) - \vartheta_n(x,t)| \leq \frac{KM^n e^{n(1-\gamma)/(1-\gamma)}}{c_t^M (a(1-\gamma)+1)}$$

The last inequality leads to the fact that the sequence $\vartheta_n$ uniformly converges to a bounded continuous function $\vartheta$ on $\mathbb{R}^k \times [0,T]$.

To prove the uniqueness, let us suppose that there are two solutions $\vartheta$ and $\vartheta^*$ of equation (2.6). Thus

$$\max_x|\vartheta(x,t) - \vartheta^*(x,t)| \leq \frac{M}{c_t^M} \int_0^T \frac{\max|\vartheta(x,\theta) - \vartheta^*(x,\theta)|}{(t-\theta)^\gamma} d\theta,$$

But $\vartheta$ and $\vartheta^*$ are bounded on $\mathbb{R}^k \times [0,T]$.

Thus there exists a constant $K > 0$ such that

$$\max_x|\vartheta(x,t) - \vartheta^*(x,t)| \leq \frac{MK^{1-\gamma}}{c_t^M (1-\gamma)^\gamma}$$

By induction, one gets:

$$\max_x|\vartheta(x,t) - \vartheta^*(x,t)| \leq \frac{KM^n e^{n(1-\gamma)/(1-\gamma)}}{c_t^M (a(1-\gamma)+1)}$$

As $n \to \infty$, we get $\vartheta(x,t) = \vartheta^*(x,t)$.

This completes the proof of the theorem.

It is easy to prove the stability of solutions of equation (2.6).

**Proof.** Let us suppose that:

$$\max_x|\phi(x)| \leq \varepsilon, \ max_x|\Psi(x,t)| \leq \varepsilon,$$

where $\varepsilon$ is a sufficiently small positive number.

Thus

$$\max_x|\vartheta(x,t)| \leq K \varepsilon + \frac{M}{c_t^M} \int_0^T \frac{\max|\vartheta(x,\theta)|}{(t-\theta)^\gamma} d\theta.$$

From the last inequality, we get

$$\max_x|\vartheta(x,t)| \leq K \varepsilon E_{1-\gamma}\left[\frac{M(1-\gamma)}{c_t^M} E_{1-\gamma}\right],$$

where $E_{\gamma}$ is the Mittag-Leffler function, defined by

$$E_{\gamma}(t) = \sum_{k=0}^\infty \frac{t^k}{\Gamma(k\gamma+1)} \quad (\text{see [22]}).$$

**Theorem 2.2** There exists a dense set $E \in C_k(\mathbb{R}^k)$ such that if
$\phi \in E$, $\Psi, b_q \in E$ for every $t \in [0,T]$, then equation (2.5) can be solved.

Moreover, this solution is unique and is continuously depending on $\phi$ and $\Psi$.

**Proof.** Let $E$ be the set of all sequences $f_n$, where

$$f_n(x) = \int \mathcal{G}(x - y, \frac{1}{n}) f(y) dy,$$

where $f \in C_p(\mathbb{R}^k)$.

It is clear that $E$ is dense in $C_p(\mathbb{R}^k)$.

Let $\vartheta(x, t, \frac{1}{n^2}, \frac{1}{n})$ be the solution of equation (2.6) with $c_1 = \frac{1}{n^2}$ and $c_2 = \frac{1}{n}$.

Set

$$u_n(x, t) = \int \mathcal{G}(x - y, \frac{1}{n} - \frac{1}{n^2}) \vartheta(y, t, \frac{1}{n^2}, \frac{1}{n}) dy.$$  

Using the semi-group property and (2.6), we find that $u_n$ satisfies the following equation

$$u_n(x, t) = \phi_n(x) + \int_0^t \Psi_n(x, \theta) d\theta + \sum_{|q| \leq m} \int_0^t b_{q,n}(x, \theta) D^q u_n(x, \theta) d\theta,$$

where $\phi_n$, $\Psi_n$ and $b_{q,n}$ are defined as in (2.7).

Hence the required result.

Notice that if the coefficients $b_q$'s do not depend on $x$ for all $|q| \leq m$, then $b_{q,n} = b_q$.

**Corollary 1.** The fractional Cauchy problem (1.4)-(1.6) can be solved, if the coefficients $a_{q,n}$ are replaced by $a_{q,n} \in E$ and the initial conditions $\phi_1$ and $\phi_2$ are replaced by $\phi_{1,n}, \phi_{2,n} \in E$, respectively (see [13, 14]). Compare also [3-10, 15, 16, 18-20].

3 A generalization of fractional differential equations

To solve equation (1.7), we consider first the following integral equation:

$$\vartheta(x, t) = \phi(x, c, t) + \int_0^t \sum_{|q| \leq m} \bar{a}_q(x, t, \theta, c, \tau) D^q \vartheta(x, \theta, c_2 - c_1 \theta) d\theta,$$  

(3.1)

Let us choose $N$ to be sufficiently large such that $\gamma < \alpha$.

From (1.8), (2.3) and using the semi-group property, we get

$$\max_{x} \left| \int_0^t \sum_{|q| \leq m} \bar{a}_q(x, t, \theta, c, \tau) D^q \vartheta(x, \theta, c_2 - c_1 \theta) d\theta \right| \leq M \int_0^t \frac{\max \{\vartheta(x, \theta)\}}{c_1^\gamma(t - \theta)^{1-\gamma}} d\theta,  \quad (3.2)$$

for some constant $M > 0$.

Using (3.2), we can prove the existence of the solution $\vartheta$ of equation (3.1).

This solution belongs to $C_p(\mathbb{R}^k \times [0,T])$ and depends continuously on $\phi$.

The steps of the proof are similar to that of theorem (2.1).

Let $\vartheta(x, t, \frac{1}{n^2}, \frac{1}{n})$ be the solution of equation (3.1) with $c_1 = \frac{1}{n^2}$, $c_2 = \frac{1}{n}$.

Let

$$u_n(x, t) = \int \mathcal{G}(x - y, \frac{1}{n} - \frac{1}{n^2}) \vartheta(y, t, \frac{1}{n^2}, \frac{1}{n}) dy.$$  

Similar to theorem (2.2), we can find that $u_n$ will represent, for every $n$, the solution to the following equation:
4 Hilfer fractional partial differential equations

The Hilfer fractional Cauchy problem (1.11), (1.12) is equivalent to:

\[ u(x, t) = \phi_n(x) + \int_0^t \sum_{\mid q \leq m} a_{q,n}(x, t, \theta) D^q u_n(x, \theta) d\theta, \]

where

\[ \phi_n(x) = \int G(x - y, \frac{1}{n}) \phi(y) dy, \]
\[ a_{q,n}(x, t, \theta) = \int G(x - y, \frac{1}{n}) a_q(y, t, \theta) dy. \]

\[ \int \sum \int \]

\[ (4.1) \]

where \( 0 < \nu \leq 1, \ 0 < \alpha \leq 1 \) (see [1, 2, 21]).

Similar to sections (2) and (3), we solve first the following equation:

\[ \theta(x, t) = \frac{t^{(\nu-1)(\alpha-1)}}{\Gamma(\nu(1-\alpha)+\alpha)} \tilde{\phi}(x, c, t) \\
+ \int_0^t (t - \theta)^{\alpha-1} \left[ \sum_{\mid q \leq m} \tilde{a}_q(x, \theta, c, t) D^q \tilde{u}(x, \theta, c_2 - c_1 \theta) \right] d\theta, \]  

\[ (4.2) \]

For \( \gamma < \alpha \), equation (4.2) has a unique solution \( \theta \in C_b (\mathbb{R}^k \times [0, T]) \).

Also the solution \( \theta \) depends continuously on \( \phi \).

Again if,

\[ \theta(x, t, \frac{1}{nt}, \frac{1}{n}) \]

is the solution of (4.2) with \( c_1 = \frac{1}{nt}, \ c_2 = \frac{1}{n} \), then

\[ u_n(x, t) = \int G(x - y, \frac{1}{n} - \frac{t}{nt}, \frac{1}{n}) \theta(y, t, \frac{1}{nt}, \frac{1}{n}) dy, \]

will satisfy the Hilfer fractional Cauchy problem (1.11), (1.12), when we replace \( a_q \) with \( a_{q,n}(x, t) = \int G(x - y, \frac{1}{n}) a_q(y, t) dy, \)

and \( \phi \) by

\[ \phi_n(y) = \int G(x - y, \frac{1}{n}) \phi(y) dy \]

Conclusion

A general singular integro-partial differential equation can be solved using the parabolic transform.

The considered equations are solved without any restrictions on the characteristic forms.

As a special case Cauchy problems are solved for general fractional partial differential equations and also for Hilfer fractional partial differential equations.

Acknowledgements

The Authors would like to thank the chair editor and the anonymous referees very much for his/her valuable comments and suggestions.

References


