Abstract:

In this work, we were able to characterize the available information about topological group and studied the widest range of group or which is called topological monoid, and we studied topological transformation monoid (M-space) and the most important characteristics and we came to define a new type of spaces called M-periodic space.

Keywords

syndetic set - topological transformation semi-group -topological semi-group -topological monoid

1. Introduction

The first principles of dynamic systems have emerged in the study of differential equations, as it is at the beginning of the twentieth century found the models of two types of types that are important, model cannot provide a definition of the system of equations him. The other model can solve the differential equations of him. Then studies have emerged in this subject without resorting to differential equations, it's called (topological dynamic), in 1955 published a book (topological dynamic) by Gohschalk and Helund [2], which contained a lot of dynamic concepts interested in symbolism of the dynamic system mean topological transformation which is a general model for dynamic systems in general. We would like to note here that the concepts that we discussed which is the Syndetic set, which have presence known in topological group where the importance of this set, written many papers that dealt with even as little to these uses set but they were not dealing with this set directly, to be released book Gohschalk and Helund (topological dynamic), where the parameters of this set and its importance became clear in the topological groups. And then expanded after that work in this area to include semi-group topological semi-group and the role of the Syndetic set where it, since fought two researchers Friedrich and Sebastian in the midst of this topic in 2013 [1].

2. Preliminary

We denoted the topological monoid by M, and topological transformation monoid by the triple (M, X, ϕ). The subset T⊆M is called right syndetic in M if there exists a compact subset K⊆M such that K1T=KT=M, and the action ϕ is said to be periodic if M(ϕ) is a right syndetic in M and then a point x∈X is said to be periodic with respect to ϕ if M(x) is a right syndetic in M. The set A is said to be invariant under M or (M-invariant) if and only if M(A)=A. (M,X,ϕ) is a topological transformation monoid (M-space) if every point of X is periodic, then X is called M-periodic space.

Definition 2.1:[4, p.14]:

A nonempty set S with a binary composition

S × S → S,

(s, t) → s . t

is called a groupoid. The composition of a groupoid is called multiplication, this multiplication is called associative if s(tr) = (st)r for all s,t,r∈S; instead of s . t we usually write st. A groupoid with associative multiplication is called a semigroup.

• S is a finite semigroup if it has only a finitely many elements.

• S is commutative semigroup, if it satisfies

xy = yx, for all x, y ∈ S
Semigroup may possess several one-sided identities. Moreover, we can always furnish them with an identity, even with a new one if they already have one.

**Definition 2.2:**[4,p.15]:
Let S be a semi group. An element e ∈ S is called a left identity of S if es = s, right identity of S if se = s, identity of S if se = es = s, for all s ∈ S. A semigroup S is monoid, if it has identity. A semi group S can have at most one identity. The identity of a monoid S is usually denoted by 1.

**Definition 2.3:**[2]:
Let X be a non-empty set. A topology on a set X is a collection T of subset of X satisfying:

1. ∅, X ∈ T
2. U ∩ V ∈ T for every U, V ∈ T
3. Uα, Vα ∈ T for every subcollection {Uα, Vα} of T.

Then (XT) is called a topological space and a subset U of X is called an open set if U ∈ T.

**3- Topological monoid:**

**Definition 3.1:**
A topological monoid is a monoid (M,+) endowed with a topology T on M, such that the binary operation µ:M×M→M is continuous.

**Examples 3.2:**

1. (R,+&U) is a topological monoid, where * is the usual multiplication and U is the usual topology on R.
2. Take the usual monoid (Z,+ & the topology T={∅,Z,0,Z(0)})
   - Is not a topological monoid since + is not continuous.

**Definition 3.3:**[1]:
Let M be a topological monoid and let T ⊆ M. Concerning an element m ∈ M, let s⁻¹T = {x ∈ M | sx ∈ T} and T s⁻¹ = {x ∈ M | xs ∈ T}. Fora subset K ⊆ M, define K⁻¹T = U{s∈K} s⁻¹T and TK⁻¹ = U{s∈K} T s⁻¹.

**Note:**
If the binary operation µ is commutative, then m⁻¹T = T m⁻¹ and K⁻¹T = TK⁻¹.

**Example 3.4:**
Take the topological monoid (M,+) and the discrete topological space since M is the set of non-negative natural number.

Let T = {xeM | xe9}, m = 2 then by the definition, 2⁻¹T = {x ∈ M | 2x ∈ T} = {0,1,2,3,4} = T ²⁻¹

Let K = {0,1,2} ⊆ M then 1⁻¹T = T and 0⁻¹T = M, hence K⁻¹T = M = TK⁻¹.

**Definition 3.5:**
A topological transformation monoid is a triple (M, X, φ) where M is a topological monoid, X is a topological space and φ: M×X→X is a continuous mapping such that:

1. φ(e, x) = x ∀x ∈ X
2. φ(m, φ(n, x)) = φ(mn, x) ∀m, n ∈ M, x ∈ X.

The space X together with the action φ of M on X is called M-space or more precisely left M-space.

**Remark 3.6:**
In above φ(m, x) we denoted by m.x, and we use the tow conditions (1),(2) will be:

1. e.x = x for all x ∈ X.
2. m.(n.x) = (mn).x for all x ∈ X and m ∈ M.

**Example 3.7:**
Take the topological monoid (R,+) and the usual topological space (R,U), defined φ:R×R→R by φ(r,t) = rt then (R,R,φ) is a topological transformation monoid.
Definition 3.8: [1]:

Let \( (M, X, \varphi) \) be a topological transformation monoid, for \( x \in X \), the set \( M_x(\varphi) = \{ m \in M \mid \varphi(m, x) = x \} \) is called the stabilizer of \( \varphi \) at \( x \), and we define \( \varphi_x : M \times X \to M \times X \) by \( \varphi_x(m, x) = (\varphi(m, y), y) \), \( \forall m \in M \) and \( y \in \varphi(m, x) \). Furthermore, we refer to \( M(\varphi) = \bigcap_{x \in X} M_x(\varphi) \) the stabilizer of \( \varphi \).

Definition 3.9:[1]:

Let \( M \) be a topological monoid , a subset \( T \subseteq M \) is called right syndetic \( M \) if there exists a compact subset \( K \subseteq M \) such that \( K^{-1}TK = M \) and \( T \) is called left syndetic if there exists a compact subset \( K \subseteq M \) such that:
\[
TK^{-1} = TK = M.
\]

Definition 3.10: [1]:

Let \( (M, X, \varphi) \) be a topological transformation monoid, then a point \( x \in X \) is said to be periodic with respect to \( \varphi \) if \( M_x(\varphi) \) is a right syndetic in \( M \).

Definition 3.11: [1]:

Let \( (M, X, \varphi) \) be a topological transformation monoid. The action \( \varphi \) is said to be periodic if \( M(\varphi) \) is a right syndetic in \( M \).

Proposition 3.12:[1]:

Let \( (M, X, \varphi) \) be a topological transformation monoid, let \( x \in X \) be a periodic point with respect to \( \varphi \). Then \( \varphi(M, x) \) is a compact. Furthermore \( x \in \varphi(M, y) \) and thus \( \varphi(M, x) = \varphi(M, y) \) \( \forall y \in \varphi(M, x) \) in a particular if \( M \) is non-empty, then \( x \in \varphi(M, x) \).

Proof:

Consider a compact subset \( K \subseteq M \) where \( KM_x(\varphi) = M = K^{-1}M_x(\varphi) \).

From the first equality, it follows that \( \varphi(M, x) = \varphi(K, x) \), wherefore \( \varphi(M, x) \) is compact. For the rest, let \( y \in \varphi(M, x) \) and let \( m \in M \) such that \( \varphi(m, x) = y \). Due to the second part of the equation above, we may find \( t \in K \) where \( tm \in M_x(\varphi) \). We observe that \( \varphi(t, y) = \varphi(tm, x) = x \), whence \( x \in \varphi(M, y) \). Therefore, \( \varphi(M, x) = \varphi(M, y) \). The final statement is an obvious consequence.

4-invariant, orbit, minimal set

Notation:

Let \( (M, X, \varphi) \) be a topological transformation monoid (M-space) and \( C \subseteq M \) and \( A \subseteq X \), we put \( C(A) = \{ \varphi(C, A) = \{ \varphi(m, x) / m \in M, x \in X \} \}

Definition 4.1:

Let \( (M, X, \varphi) \) be a topological transformation monoid (M-space) and \( A \subseteq X \), the set \( A \) is said to be invariant under \( M \) or \( M \)-invariant if and only if \( M(A) = A \).

Definition 4.2:

Let \( X \) be M-space and \( x \in X \), the orbit of \( x \) under \( M \) (M-orbit of \( x \)) is denoted by \( M(x) \) or \( M_x \) such that \( M_x = \{ mx / m \in M \} \). The orbit closure of \( x \) under \( M \) is denoted by \( \overline{M_x} \).

Definition 4.3:

Let \( X \) be M-space and \( A \) be subset of \( X \) and \( S \subseteq M \), then the set \( A \) is said to be minimal under \( S \) (S-minimal) provided that \( A \) is an orbit closure under \( S \) and \( A \) does not contain properly an orbit closure under \( S \).

If \( S = M \), we often use more colorful phrase “minimal orbit closure” in preference of minimal set.

Proposition 4.4:

Let \( X \) be M-space, then the following statements are valid: 1) If \( x \in X \), then the orbit of \( x \) under \( M \) is the least \( M \)-invariant subset of \( X \) which contains the point \( x \). 2) If \( x \in X \), then the orbit closure of \( x \) under \( M \) is the least closed \( M \)-invariant which contain \( x \).

Proof:

1) Let \( x \in X \) and \( M_x \) be the orbit of \( x \) under \( M \), T.P. \( M(M(x)) = M(x) \).
\[
M(M(x)) = M(\{mx / m \in M\}) = \{m_2(m_1x) / m_2, m_1 \in M\}
\]

But \( m_2, m_1 \in M \) such that \( M \) is a monoid, then
M(M(x))= (mx | m ∈ M} = M(x).

Now T.P. M(x) is the least M-invariant subset of X.

Suppose there exist A is M-invariant subset of X, such that x ∈ A and A ⊆ M(x) . We have A is M-invariant, M(A) = A, thus M(A) ⊆ M(x) but M(x) ⊆ M(A), s.t. x ∈ A ⇒ M(A) = M(x) ⇒ A = M(x). Hence M(x) is the least one .

2) Let x ∈ X, Mx is an orbit-closure of x under M . To prove Mx is M-invariant . We have Mx is M-invariant , by proposition () Mx is M-invariant . Now to prove Mx is the least closed M-invariant subset of X (s.t x ∈ Mx ) . Suppose A is a closed M-invariant subset of X , which contains x . Since x ∈ A, then Mx ⊆ M(A) ⇒ Mx ⊆ M(A) ⊆ Mx (A is M-invariant). Hence Mx is the least closed M-invariant .

5- M-periodic space:

Definition 5.1:

Let (M,X,ϕ) be a topological transformation monoid (M-space) if every point of X is periodic , then X is called M-periodic space .

Examples 5.2:

i- Suppose there exist a topological transformation monoid (N,R,ϕ) such that ϕ :N×R→R defined by ϕ(n,r)=r , then N×N , ∀ x∈X . T.P. N is right syndetic .

Since there exist {0} a compact subset of N w.r.t Discreet topology on N such that {0}¬1+N={0}+N=N , thus N is right syndetic. i.e. ∀x∈R , x is a periodic point . Hence R is N-periodic space .

ii- At the same example (i) if we take Q(n,r)=n+r .

⇒ N_{x}={m∈N | mx=x} ={m∈N | m+x=x}={0} , ∀ x∈R , then

\#H=N \neq N and H^{-1}+N=x N except N itself . Put N is not compact because its not finite set . Hence there is no periodic point at this space .

Proposition 5.3:

If X is M-space , then we can defined a relation R on X such that R={(x,y)∈X×X/ y∈Mx} , R is equivalence relation on X .

Proof:

Let e be the identity element in M .

i- Since e ∈ M ⇒ e×x = x ∈ Mx = x ∈ R .

ii- Let (x,y) ∈ R , y ∈ Mx ⇒ Mx ⊆ Mx ⇒ x ∈ Mx ⇒ (x,y) ∈ R .

iii- Let (x,y) , (y,z) ∈ R ⇒ y ∈ Mx ⇒ Mx ⊆ My and z ∈ My ⇒

Mx = My ⇒ Mx = Mx ⇒ z ∈ Mx ⇒ (x,z) ∈ R .

Now we defined [x]={y ∈ X / y ∈ Mx}= Mx , ∀ x∈X .

Now we can define the set of all orbits which denoted by X/M

(i.e. X/M =X/R) . X/M ={ Mx | x∈X } .

Let π : X→ X/M denoted the canonical mapping (quotient mapping ), that π(x)= Mx , then X/M endowed with quotient topology

T'=∪∈X/M | π¹'(U) is open in X} is called M-orbit space of X .

Proposition 5.4:

Let X be M-periodic space , then the following statements are valid :

1) If x ∈ X and y ∈ Mx , then Mx = My .

2) The class of all orbits under M is a partition of X .

3) If x ∈ X and y ∈ Mx , then Mx ⊆ Mx .
4) The class of all orbit closure under M is a covering of X.

Proof:

1) Let \( x \in X \), which is a periodic point and let \( y \in M(x) \), by proposition

\[ M_x = M_y. \]

2) \( L = \{ M_x \mid M_x \text{ is the orbit of } x \text{ under } M \} \). T.P. A is a partition of X.

We take \( \forall x \in X \), \( x \) is a periodic point, it means that \( M_x \cap M_y = \emptyset \), \( \forall x, y \in X \) such that if \( M_x \cap M_y \neq \emptyset \), then \( M_x = M_y \).

Now we have \( x \in M_x, \forall \ x \in X \) s.t \( e \in M \) and \( M_x \subseteq X \), \( \forall x \in X \). Hence \( uM = X \), then A is a partition of X.

3) Assume that \( y \in \overline{M_x} \), then \( M_y \cap \overline{M_x} = \overline{M_x} \). Hence \( \overline{M_y} \subset \overline{M_x} \).

4) Let A be the class of all orbit closures under M, now to prove \( X \subseteq \bigcup_{x \in X} \overline{M_x} \). Let \( x \in X \), then \( x \in \overline{M_x} \Rightarrow x \in \bigcup_{x \in X} \overline{M_x} \Rightarrow X \subseteq \bigcup_{x \in X} \overline{M_x} \). Hence \( \{ \overline{M_x} \mid x \in X \} \) is covering.

**Proposition 5.5:**

If \( X \) is M-periodic space and \( X \) is T\(_2\)-space, then \( M_x \) is closed set. Furthermore, \( M_x \) is minimal for all \( x \in X \).

**Proof:**

Suppose \( X \) is M-periodic space, \( x \) is periodic point for all \( x \in X \), then by proposition (2.12) \( M_x \) is a compact subset of \( X \). But \( X \) is T\(_2\)-space, then \( M_x \) is closed subset of \( X \), thus \( M_x = \overline{M_x} \).

Suppose there exists properly an orbit closure under M (i.e. there exists \( \overline{M_y} \subseteq M_x \), but \( M_y = M_x \)). Hence \( M_x \) is minimal set.

**Proposition 5.6:**

Let \( X \) be M-periodic space and T\(_2\)-space, let \( A \subseteq M \), then \( A \) is M-invariant set if and only if, for every \( y \in A \), we have \( \overline{M_y} \subseteq A \).

**Proof:**

\( \Rightarrow \) Suppose \( A \) is M-invariant set, thus \( M(A) = A \). Let \( y \in A \), then \( M_y \subseteq M(A) = A \Rightarrow y \) is periodic point \( \Rightarrow M_y \) is closed. Hence \( M_y = \overline{M_y} \subset A \).

\( \Leftarrow \) Suppose that for each \( y \in A \), we have \( \overline{M_y} \subset A \), to prove that \( A \) is M-invariant. Clear \( A \subseteq M(A) \) such that \( e \in M \), since \( \overline{M_y} \subset A \) for each \( y \in A \), then \( U_{y \in A} \overline{M_y} \subset A \Rightarrow M(A) = \bigcup_{y \in A} \overline{M_y} \subseteq A \). Hence \( M(A) = A \).

**Proposition 5.7:**

Let \( X \) be M-periodic space and T\(_2\)-space, and let \( A \) be subset of \( X \), then \( A \) is minimal orbit-closure under M if and only if

- \( A \neq \emptyset \), and \( \overline{M_x} = A \), for each \( x \in A \).

**Proof:**

Let \( X \) be M-periodic space and A be subset of \( X \),

\( \Rightarrow \) suppose \( A \) is minimal orbit-closure under M, by proposition (2.12) \( A \) is closed M-invariant and \( A \) is minimal with respect to this property \( \Rightarrow \) by proposition (2.12) \( \overline{M_x} \subseteq A \), for each \( x \in A \), but \( \overline{M_x} \) is closed M-invariant \( \Rightarrow A \subseteq \overline{M_x} \Rightarrow \overline{M_x} = A \), for each \( x \in A \).

\( \Leftarrow \) suppose \( A \neq \emptyset \), and \( \overline{M_x} = A \), for each \( x \in A \) to prove \( A \) is minimal set.

We must prove \( A \) do not contain properly any orbit-closure.

Suppose \( \overline{M_y} \subset A \Rightarrow y \in A \Rightarrow \overline{M_y} = A \Rightarrow A \) is minimal set.

**Proposition 5.8:**

Let \( X \) be M-periodic space and T\(_2\)-space, then the following statement are pairwise equivalent.

- The class of all orbit-closure under M is a partition of \( X \).
- If \( x \in X \) and if \( y \notin \overline{M_x} \), then \( \overline{M_x} = \overline{M_y} \).
- Every orbit-closure under M is minimal under M.
- The class of all minimal orbit-closure under M is a covering of \( X \).
Proof:
Let $X$ be $M$-periodic space and $T_2$-space

i) Since $\overline{M_x} = M_x$ for all $x \in X$ (by proposition2.11) $\Rightarrow \overline{M_x} \cap \overline{M_y} = \emptyset$ and $U_{x \in X} \overline{M_x} = X \Rightarrow$ The class of all orbit-closure under $M$ is a partition of $X$

ii) Clear

iii) Let $x \in X \Rightarrow x$ is periodic point.

To prove $\overline{M_x}$ is minimal set

By using (ii) then for every $y \in \overline{M_x}$, then $\overline{M_x} = \overline{M_y}$

Hence $\overline{M_x}$ is minimal.

iv) Let $\Omega = \{ \overline{M_x} : x \in X \}$ be The class of all minimal orbit-closure under $M$, now to prove $X \subseteq U_{x \in X} \overline{M_x}$.

Let $x \in X$, then $x \in \overline{M_x} \Rightarrow x \in U_{x \in X} \overline{M_x} \Rightarrow X \subseteq U_{x \in X} \overline{M_x}$. Hence

$\Omega = \{ \overline{M_x} : x \in X \}$ is covering.

References