On Two Preys and Two Predators Model with Prey Migration and Predator Switching

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Abstract
The present paper deals with model which describes the interaction of two prey species with two predators. The prey species have the ability of group defense and is assumed to live in two distinct habitats and the predators’ species have the tendency of switching between the habitats. The model is analyzed regarding boundedness of the positive solutions, and stability of equilibrium points. Numerical simulations are carried out to support analytical findings.

Keywords: Prey, Predator, group defense, Switching mechanism, Habitats

1. INTRODUCTION:
In the environment of predator and prey, the predator prefers to feed itself in a habitat for a specified time and changing to another habitat. This phenomenon is called switching. Models that involve the interaction between one predator and two prey species with the switching have been proposed by Tansky [8], Prajneshu and Holgate [7], Khan et al. [5]. And model that involve the interaction between two predator and two prey species with the switching have been proposed by by Khan et al. [6].

In [4], R. Bhattacharyya, and B. Mukhopadhyay studied prey-predator eco-system models under the assumptions of seasonal migration of the prey population and switching mechanism of the predator. It is known that, the prey population is assumed to live in two different habitats. They studied the impact of predator switching in two different contexts. i. e. with and without group defense among the prey. The models are studied for the values n=1 and 2 of the switching index and the called model are studied for n≠1, in [2] and [3] with and without group defense among the prey respectively.

The present paper deals with two prey and two predators model, with prey group defense. The prey species is assumed to live in two different habitats and the predators’ species have the tendency of switching between the habitats. The models are studied for the values n=1 of the switching index.

2. THE MATHEMATICAL MODEL:
The two prey and two predators model, with prey group defense and the prey species is assumed to live in two distinct habitats and the predators species have the tendency of switching between the habitats is of the form:

\[
\begin{align*}
\dot{x}_1 &= x_1 \left[ g_1 \left( 1 - \frac{x_1}{k_1} \right) - \frac{a_1x_1y_1}{x_1^2 + x_2^2} - \frac{a_2x_1y_2}{x_1 + x_2^2} \right] x_1^n + x_2^n > 0, \\
\dot{x}_2 &= x_2 \left[ g_2 \left( 1 - \frac{x_2}{k_2} \right) - \frac{\beta_1x_1y_1}{x_1^n + x_2^n} - \frac{\beta_2x_2y_2}{x_1^n + x_2^n} \right] x_1^n + x_2^n > 0, \tag{2.1}
\end{align*}
\]

\[
\begin{align*}
\dot{y}_1 &= -\mu_1 + \frac{\delta_1x_1y_1^n}{x_1^n + x_2^n} + \frac{\delta_2x_1^n}{x_1^n + x_2^n} y_1, \quad y_1 \geq 0. \\
\dot{y}_2 &= -\mu_2 + \frac{y_1^n}{x_1^n + x_2^n} + \frac{y_2^n}{x_1^n + x_2^n} y_2, \quad y_2 \geq 0.
\end{align*}
\]

with n=1, the above system become

\[
\begin{align*}
\dot{x}_1 &= x_1 \left[ g_1 \left( 1 - \frac{x_1}{k_1} \right) - \frac{a_1x_1y_1}{x_1 + x_2} - \frac{a_2x_1y_2}{x_1 + x_2^2} \right] x_1 + x_2 > 0, \\
\dot{x}_2 &= x_2 \left[ g_2 \left( 1 - \frac{x_2}{k_2} \right) - \frac{\beta_1x_1y_1}{x_1 + x_2} - \frac{\beta_2x_2y_2}{x_1 + x_2^2} \right] x_1 + x_2 > 0, \tag{2.2}
\end{align*}
\]

\[
\begin{align*}
\dot{y}_1 &= -\mu_1 + \frac{\delta_1x_1y_1}{x_1 + x_2} + \frac{\delta_2x_1^n}{x_1 + x_2^n} y_1, \quad y_1 \geq 0. \\
\dot{y}_2 &= -\mu_2 + \frac{y_1^n}{x_1 + x_2} + \frac{y_2^n}{x_1 + x_2} y_2. \quad y_2 \geq 0.
\end{align*}
\]
where $x_1$ and $x_2$ denote prey density in two the habitats, $y_1$ and $y_2$ the denote predators density. The prey population is assumed to grow logistically with a specific growth rate $g_1$ and environmental carrying capacity $k_i, \alpha_i, i = 1, 2$ represent the predation rate in the two habitats; $\delta_i, i = 1, 2 (\gamma_i, i = 1, 2)$ are the rate of conversion of prey to predator $y_1(y_2)$.

The predation functions $\frac{\alpha_1 y_1 y_2}{x_1 + x_2}, i = 1, 2$ and $\frac{\alpha_2 y_2 y_1}{x_1 + x_2}, i = 1, 2$, model switching behavior of the predators $y_i, i = 1, 2$ in the realm of prey group defense, namely that, there will be less predation in the habitat having larger prey density.

The mortality rate of $y_i$ predator divided by the conversion of prey predator $y_1$ equal to the rate of death rate of the predator $y_2$ divided by the conversion of prey predator $y_2$ rate.

3. BOUNDEDNESS OF THE POSITIVE SOLUTIONS

Let $D := \{(x_1, x_2, y_1, y_2) \in \mathbb{R}^4, x_i \in (0, k_i), y_i > 0, i = 1, 2 \}$. (3.1)

Lemma 1. If $\delta_1 + \delta_2 \leq \alpha_1 + \beta_1 y_1 + y_2 \leq \alpha_2 + \beta_2$, then all trajectories of (2.1) with initial conditions from $D$ are bounded.

Proof: Choose the function:

$$u(t) = x_1(t) + x_2(t) + y_1(t) + y_2(t).$$

and calculate the derivative of $u(t)$ along the solution of (2.1) we have:

$$\dot{u} = x_1 g_1 \left(1 - \frac{x_1}{k_1} \right) + x_2 g_2 \left(1 - \frac{x_2}{k_2} \right) - y_1 \mu_1 + \frac{(\delta_1 - \beta_1) x_1 x_2 y}{x_1 + x_2} + \frac{(\delta_2 - \beta_2) x_2 x_1 y}{x_1 + x_2}$$

$$= x_1 g_1 \left(1 - \frac{x_1}{k_1} \right) + x_2 g_2 \left(1 - \frac{x_2}{k_2} \right) - (y_1 \mu_1 + y_2 \mu_2) + \frac{x_1 x_2}{x_1 + x_2} [(\delta_1 + \delta_2 - \alpha_1 - \beta_1) y_1 + (y_1 + y_2 - \alpha_2 - \beta_2) y_2]$$

For a positive constant $\rho \leq \max(\mu_1, \mu_2)$, we have:

$$\dot{u} + \rho u = x_1 g_1 \left(1 - \frac{x_1}{k_1} \right) + x_2 g_2 \left(1 - \frac{x_2}{k_2} \right) + (y_1 (\rho - \mu_1) + y_2 (\rho - \mu_2)) + \frac{x_1 x_2}{x_1 + x_2} [(\delta_1 + \delta_2 - \alpha_1 - \beta_1) y_1 + (y_1 + y_2 - \alpha_2 - \beta_2) y_2]$$

$$\dot{u} + \rho u \leq \frac{x_1}{k_1} (k_1 g_1 - g_1 x_1 + k_1 \rho) + \frac{x_2}{k_2} (k_2 g_2 - g_2 x_2 + k_2 \rho) <$$

$$< \frac{x_1}{k_1} (k_1 g_1 + k_1 \rho) + \frac{x_2}{k_2} (k_2 g_2 + k_2 \rho) <$$

$$< k_1 (g_1 + \rho) + k_2 (g_2 + \rho)$$

This leads to $0 \leq u(t) \leq \frac{u(0)}{\rho} + u(0) e^{-\rho t}$, and for $t \to \infty, 0 \leq u(t) \leq \frac{u(0)}{\rho}$ where $\alpha = k_1 (g_1 + \rho) + k_2 (g_2 + \rho)$.

Hence, we obtain that all the positive solutions of the system (2.2) with initial conditions $(\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2) \in D$, and satisfy $\delta_1 + \delta_2 \leq \alpha_1 + \beta_1 y_1 + y_2 \leq \alpha_2 + \beta_2$, are bounded.

4. STABILITY OF EQUILIBRIA:

In this section, the existence of the equilibrium points of the system (2.2) and the stability near them are investigated. We obtain equilibrium points by setting derivative terms to zero. The system has the following nonnegative equilibrium points:

I. The prey population grows in the absence of predator, while the predator dies in the absence of prey, then the equilibrium points $(k_1, 0, 0, 0)$ and $(0, k_2, 0, 0)$ always exists. The eigenvalues of the two characteristic equations of the variational matrix of the system (2.2) at the equilibrium points $(k_1, 0, 0, 0)$ and $(0, k_2, 0, 0)$ are:

$$\lambda = -g_1 < 0, \quad \lambda = g_2 > 0, \quad \lambda = -\mu_1 < 0, \quad \lambda = -\mu_2 < 0 \text{ and } \lambda = g_1 - g_2 < 0.$$ 

Which mean that the two equilibrium points are saddle points (unstable).

II. The prey population grows in the absence of predators, while the predators die in the absence of prey, then the equilibrium point $(k_1, k_2, 0, 0)$ always exists. The characteristic equation of the variational matrix of the system (2.2) at the equilibrium
point \((k_1, k_2, 0, 0)\) is written as follows:

\[
(\lambda + g_1)(\lambda + g_2) \left( \frac{(\delta_1 + \delta_2)k_1k_2}{k_1 + k_2} - \mu_1 - \lambda \right) \left( \frac{(\gamma_1 + \gamma_2)k_1k_2}{k_1 + k_2} - \mu_2 - \lambda \right) = 0,
\]

\[
\lambda = -g_1 < 0, \quad \lambda = -g_2 < 0.
\]

\[
\lambda = \frac{(\delta_1 + \delta_2)k_1k_2}{k_1 + k_2} - \mu_1, \quad \lambda = \frac{(\gamma_1 + \gamma_2)k_1k_2}{k_1 + k_2} - \mu_1
\]

Hence, the equilibrium \((k_1, k_2, 0, 0)\) will be locally stable to small perturbations if and only if it satisfies the following conditions:

\[
\begin{cases}
(\delta_1 + \delta_2)k_1k_2 < \mu_1 \\
(\gamma_1 + \gamma_2)k_1k_2 < \mu_1
\end{cases}
\]

(4.1)

III. The interior equilibrium point

We will study the interior equilibrium point of the system (2.2) given by \((x_k, y_k, y_{k+1}, y_{k+2})\), where:

\[
\begin{align*}
\bar{x}_1 &= \frac{\mu_1 (1 + \bar{x})}{\delta_1 + \delta_2} = \frac{\mu_2 (1 + \bar{x})}{\gamma_1 + \gamma_2}, \\
\bar{x}_2 &= \frac{\mu_1 (1 + \bar{x})}{\bar{x}(\delta_1 + \delta_2)} = \frac{\mu_2 (1 + \bar{x})}{\bar{x}(\gamma_1 + \gamma_2)}, \\
\bar{y}_1 &= \frac{\beta_2 g_1 x(x - \frac{x}{k_2}) - \alpha_2 g_2 x(\frac{x}{k_1} - \frac{x}{k_2})}{(\alpha_1 \beta_2 - \alpha_2 \beta_1)x} (1 + \bar{x}) \quad (4.2) \\
\bar{y}_2 &= \frac{\alpha_1 g_2 x(\frac{x}{k_2} - \beta_1 g_1 x(\frac{x}{k_1})}{(\alpha_1 \beta_2 - \alpha_2 \beta_1)x} (1 + \bar{x})
\end{align*}
\]

\[
\frac{\mu_1}{\delta_1 + \delta_2} = \frac{\mu_2}{\gamma_1 + \gamma_2}
\]

and \(\bar{x} = \frac{x_1}{x_k}\) is real positive root of the following equation:

\[
\beta_2 g_2 k_2 \mu_1 \bar{x}^3 - \beta_2 g_2 k_2 [k_1(\delta_1 + \delta_2) - \mu_1] \bar{x}^2 + \alpha_2 g_2 k_1 [k_2(\delta_1 + \delta_2) - \mu_1] \bar{x} - \alpha_2 g_2 k_1 \mu_1 = 0. \quad (4.3)
\]

By simple calculator the characteristic equation of the variation matrix of the system (2.1) at the point \((\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2)\) is written as follows:

\[
\begin{vmatrix}
A_1 - \lambda & -A_2 \bar{x} & \alpha_1 M \alpha_2 M \\
-A_1 \bar{x} & A_1 - \frac{g_2 \bar{x}^2}{k_2} - \lambda & \beta_1 M \beta_2 M \\
R_1 & R_1 \bar{x}^2 & -\lambda \\
R_2 & R_2 \bar{x}^2 & 0 - \lambda
\end{vmatrix} = 0,
\]

\[
\begin{align*}
A_1 &= g_2 (1 - \frac{\bar{x}}{k_2}) \frac{1}{1 + \bar{x}} \\
\frac{\bar{x}}{1 + \bar{x}} & M = \frac{\bar{x}}{1 + \bar{x}}, \\
R_1 &= \frac{g_2 (1 - \frac{\bar{x}}{k_2}) \bar{y}_2}{(1 + \bar{x})^2}, \\
R_2 &= \frac{(\gamma_1 + \gamma_2) \bar{y}_2}{(1 + \bar{x})^2}
\end{align*}
\]

which leads to the following eigenvalue equation:

\[
\begin{vmatrix}
A_1 - \lambda & -A_2 \bar{x} & \alpha_1 M \alpha_2 M \\
-A_1 \bar{x} & A_1 - \frac{g_2 \bar{x}^2}{k_2} - \lambda & \beta_1 M \beta_2 M \\
R_1 & R_1 \bar{x}^2 & -\lambda \\
R_2 & R_2 \bar{x}^2 & 0 - \lambda
\end{vmatrix} = 0,
\]

\[
\begin{align*}
A_1 &= g_2 (1 - \frac{\bar{x}}{k_2}) \frac{1}{1 + \bar{x}} \\
\frac{\bar{x}}{1 + \bar{x}} & M = \frac{\bar{x}}{1 + \bar{x}}, \\
R_1 &= \frac{g_2 (1 - \frac{\bar{x}}{k_2}) \bar{y}_2}{(1 + \bar{x})^2}, \\
R_2 &= \frac{(\gamma_1 + \gamma_2) \bar{y}_2}{(1 + \bar{x})^2}
\end{align*}
\]
\[(\lambda^3 + a\lambda^2 + b\lambda + c)\lambda = 0,\]

\[\Rightarrow \lambda = 0, \quad \text{or} \lambda^3 + a\lambda^2 + b\lambda + c = 0 (4.5)\]

\[
\bar{a} = \sum_{i=1}^{2} \left( \frac{g_i x_i}{k_i} - A_i \right), \\
\bar{b} = \frac{g_1 x_1 g_2 x_2}{k_1 k_2} - \sum_{i=1}^{2} \left[ MR_i (\alpha_i + x^2 \beta_i) - \frac{g_i x_i A_i}{k_i} \right] c = \sum_{i=1}^{2} MR_i \left[ \left( A_1 (x + 1) - \frac{g_2 x_2}{k_2} \right) \alpha_i + \left( A_2 (x + 1) - \frac{g_2 x_2}{k_1} \right) \beta_i \right]
\]

The Routh-Hurwitz stability criteria for the third order system is \(\bar{a} > 0, \bar{b} > 0\) and \(\bar{a} \bar{b} > c\), see [1]. Hence, the equilibrium \((x_1, x_2, y_1, y_2)\) will be locally stable but not asymptotically stable because one of the eigenvalues is zero (4.4), if it satisfies the following conditions:

\[
\begin{align*}
\sum_{i=1}^{2} \left( \frac{g_i x_i}{k_i} - A_i \right) & > 0, \quad (4.6a) \\
\sum_{i=1}^{2} R_i \left[ \left( \frac{g_i x_i x^2}{k_i} - A_1 (x + 1) \right) \alpha_i + \left( \frac{g_i x_i x^2}{k_1} - A_2 (x + 1) \right) \beta_i \right] & > 0, \quad (4.6b) \\
\sum_{i=1}^{2} MR_i \left[ \left( A_2 - A_1 x - \frac{g_1 x_i}{k_1} \right) \alpha_i + \left( A_1 x - A_2 - \frac{g_2 x_i x^2}{k_2} \right) x \beta_i \right] & > 0, \quad (4.6c)
\end{align*}
\]

5. NUMERICAL SIMULATIONS:

In this section we aim to see the effect of various parameters on the stability equilibrium points. The system (2.1) with the parameters in Table(I), has \((0.9515, 1.0537, 0.0135, 0.6058)\), as the interior equilibrium point, which is locally stable as shown in fig(1). The equilibrium point \((k_1, 0, 0, 0)\) is unstable as shown in fig(2), while the equilibrium points \((0, k_2, 0, 0)\) and \((0, 0, k_2, 0)\) are unstable.

<table>
<thead>
<tr>
<th>(i)</th>
<th>(k_i)</th>
<th>(g_i)</th>
<th>(\mu_i)</th>
<th>(\delta_i)</th>
<th>(\gamma_i)</th>
<th>(\alpha_i)</th>
<th>(\beta_i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.1</td>
<td>1.6</td>
<td>0.4</td>
<td>0.5</td>
<td>0.7</td>
<td>0.8</td>
<td>0.4</td>
</tr>
<tr>
<td>2</td>
<td>1.2</td>
<td>1.2</td>
<td>0.6</td>
<td>0.3</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
</tr>
</tbody>
</table>
The system (2.1) with the parameters in Table (II), has no interior equilibrium point. The equilibrium point \((k_1, k_2, 0, 0)\) is locally asymptotically stable as shown in fig(3), while the equilibrium points \((k_1, 0, 0, 0)\) and \((0, k_2, 0, 0)\) are unstable.

Table (II)

<table>
<thead>
<tr>
<th>(i)</th>
<th>(k_i)</th>
<th>(g_i)</th>
<th>(\mu_i)</th>
<th>(\delta_i)</th>
<th>(\gamma_i)</th>
<th>(\alpha_i)</th>
<th>(\beta_i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.1</td>
<td>1.6</td>
<td>0.6</td>
<td>0.5</td>
<td>0.7</td>
<td>0.8</td>
<td>0.4</td>
</tr>
<tr>
<td>2</td>
<td>1.18</td>
<td>1.2</td>
<td>0.9</td>
<td>0.3</td>
<td>0.5</td>
<td>0.7</td>
<td>0.5</td>
</tr>
</tbody>
</table>
6. CONCLUSION:

In this paper we study the interaction of two prey species with two predators. The prey species have the ability of group defense and is assumed to live in two distinct habitats and the predators’ species have the tendency of switching between the habitats. Analytical, it is observed that:

all the trajectories of the system (2.1), under the condition (3.1) are bounded and the system has at most four equilibrium points, \((k_1,0,0,0)\), \((0,k_2,0,0)\) are unstable, \((k_1,k_2,0,0)\) is locally asymptotically stable if it satisfy (4.1), and the point \((\tilde{x}_1,\tilde{x}_2,\tilde{y}_1,\tilde{y}_2)\), is locally stable if it is satisfy (4.6). The point \((\tilde{x}_1,\tilde{x}_2,\tilde{y}_1,\tilde{y}_2)\) may be not exists, and. Numerically two examples of system are given, one of them has four equilibrium points, \((k_1,0,0,0)\), \((0,k_2,0,0)\), \((k_1,k_2,0,0)\) are unstable and the point is \((\tilde{x}_1,\tilde{x}_2,\tilde{y}_1,\tilde{y}_2)\) is locally stable. The other has only three equilibrium points, \((k_1,0,0,0)\), \((0,k_2,0,0)\), are unstable as we know and \((k_1,k_2,0,0)\) is locally asymptotically stable.

REFERENCES


