Qualitative behavior for a class of nonlinear difference equation

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Abstract

We study the Stability, the periodicity, the boundedness and the global stability of the positive solutions of the following nonlinear difference equation

\[ x_{n+1} = a + \sum_{i=0}^{k} c_i x_{n-i} l b x_{n-i} + dx_{n-k}, \]

where the initial conditions \( x_i, x_{i+1}, \ldots, x_0 \) are arbitrary positive real numbers such that \( r = \max\{i, k\} \) where \( i, r \in \{0,1,\ldots\} \) and \( a, b, d, c_i \) are positive constants.

Keywords

Difference equation, Stability, Periodicity, boundedness, global Stability.

1. INTRODUCTION

In this paper we deal with some properties of the solutions of the difference equation

\[ x_{n+1} = a + \sum_{i=0}^{k} c_i x_{n-i} l b x_{n-i} + dx_{n-k}, \quad n = 0,1,2,\ldots, \tag{1} \]

where the initial conditions \( x_i, x_{i+1}, \ldots, x_0 \) are arbitrary positive real numbers such that \( r = \max\{i, k\} \) where \( i, r \in \{0,1,\ldots\} \) and \( a, b, d, c_i \) are positive constants. There is a class of nonlinear difference equations, known as the rational difference equations, each of which consists of the ratio of two polynomials in the sequence terms in the same form. There has been a lot of work concerning the global asymptotic of solutions of rational difference equations [4], [5], [6], [8], [9], [13] and [14].

Many researchers have investigated the behavior of the solution of difference equation for example:

Amleh et al. [2] has studied the global stability, boundedness and the periodic character of solutions of the equation

\[ x_{n+1} = \alpha + \frac{x_{n-1}}{x_n}. \]

Saleh et al. [17] has studied the global asymptotic stability of solutions of the equation

\[ x_{n+1} = \alpha + \frac{y_n}{y_{n-k}}. \]

Our aim in this paper is to extend and generalize the work in [1], [2], [3], [11], [12], [16] and [17]. That is, we will investigate the global behavior of (1) including the asymptotical stability of equilibrium points, the existence of bounded solution, the existence of period two solution of the recursive sequence of Eq. (1).

Now we recall some well-known results, which will be useful in the investigation of (1) and which are given in [10].

Let I be an interval of real numbers and let

\[ F : I^{k+1} \to I, \]

where \( F \) is a continuous function. Consider the difference equation

\[ y_{n+1} = F(y_n, y_{n-1}, \ldots, y_{n-k}), \quad n = 0,1,2,\ldots, \tag{2} \]
With the initial condition \( y_{-k}, y_{-k+1}, \ldots, y_0 \in I \).

### 1.1 Definition (Equilibrium Point)

A point \( \bar{y} \in I \) is called an equilibrium point of Eq. (2) if

\[
\bar{y} = f(\bar{y}, \bar{y}, \ldots, \bar{y})
\]

That is, \( y_n = \bar{y} \) for \( n \geq 0 \), is a solution of Eq. (2), or equivalently, \( \bar{y} \) is a fixed point of \( f \).

### 1.2 Definition (Stability)

Let \( \bar{y} \in (0, \infty) \) be an equilibrium point of Eq. (2). Then

1) An equilibrium point \( \bar{y} \) of Eq. (2) is called locally stable if for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that, if \( y_{-r}, y_{-r+1}, \ldots, y_0 \in (0, \infty) \) with

\[
|y_{-r} - \bar{y}| + |y_{-r+1} - \bar{y}| + \ldots + |y_0 - \bar{y}| < \delta,
\]

then \( |y_n - \bar{y}| < \varepsilon \) for all \( n \geq -r \).

2) An equilibrium point \( \bar{y} \) of Eq. (2) is called locally asymptotically stable if \( \bar{y} \) is locally stable and there exists \( \gamma > 0 \) such that, if \( y_{-r}, y_{-r+1}, \ldots, y_0 \in (0, \infty) \) with

\[
|y_{-r} - \bar{y}| + |y_{-r+1} - \bar{y}| + \ldots + |y_0 - \bar{y}| < \gamma,
\]

then

\[
\lim_{n \to \infty} y_n = \bar{y}.
\]

3) An equilibrium point \( \bar{y} \) of Eq. (2) is called a global attractor if for every \( y_{-r}, y_{-r+1}, \ldots, y_0 \in (0, \infty) \) we have

\[
\lim_{n \to \infty} y_n = \bar{y}.
\]

4) An equilibrium point \( \bar{y} \) of Eq. (2) is called globally asymptotically stable if \( \bar{y} \) is locally stable and a global attractor.

5) An equilibrium point \( \bar{y} \) of Eq. (2) is called unstable if \( \bar{y} \) is not locally stable.

### 1.3 Definition (Permanence)

Eq. (2) is called permanent if there exists numbers \( m \) and \( M \) with \( 0 < m < M < \infty \) such that for any initial conditions \( y_{-r}, y_{-r+1}, \ldots, y_0 \in (0, \infty) \) there exists a positive integer \( N \) which depends on the initial conditions such that

\[
m \leq y_n \leq M \quad \text{forall} \quad n \geq -N.
\]

### 1.4 Definition (Periodicity)

A sequence \( \{x_n\}_{n=-r}^{\infty} \) is said to be periodic with period \( P \) if \( x_{n+P} = x_n \) for all \( n \geq -r \). A sequence \( \{x_n\}_{n=-r}^{\infty} \) is said to be periodic with prime period \( P \) if \( P \) is the smallest positive integer having this property.

The linearized equation of Eq. (2) about the equilibrium point \( \bar{x} \) is defined by the equation

\[
z_{n+1} = \sum_{i=0}^{k} p_i z_{n-i}, \quad (3)
\]

where

\[
p_i = \frac{\partial F(\bar{y}, \bar{y}, \ldots, \bar{y})}{\partial y_{n-i}}, \quad i = 0, 1, \ldots, k.
\]
The characteristic equation associated with Eq. (3) is

\[ \lambda^{k+1} - p_0\lambda^k - p_1\lambda^{k-1} - \ldots - p_{k-1}\lambda - p_k = 0. \tag{4} \]

1.1 Theorem [7] Let \([a, b]\) be an interval of real numbers and assume that

\[ f : [a, b]^{k+1} \to [a, b] \]

Is a continuous function satisfying the following properties:

1) \(f(x_1, x_2, \ldots, x_{k+1})\) is non-increasing in the first \(k\) terms for each \(x_{k+1}\) in \([a, b]\) and non-decreasing in the last term for each \(x_i\) in \([a, b]\) for all \(i = 1, 2, \ldots, k\).

2) If \((m, M) \in [a, b] \times [a, b]\) is a solution of the system

\[ M = f(M, m, m, \ldots, m, M) \quad \text{and} \quad m = f(M, M, M, \ldots, M, m), \]

implies

\[ m = M. \]

1.2 Theorem [10]. Assume that \( F \) is a \(C^1\) function and let \( \bar{y} \) be an equilibrium point of Eq. (2). Then the following statements are true:

1) If all roots of Eq. (4) lie in the open unit disk \(|\lambda| < 1\), then the equilibrium point \( \bar{y} \) is locally asymptotically stable.

2) If at least one root of Eq. (4) has absolute value greater than one, then the equilibrium point \( \bar{y} \) is unstable.

3) If all roots of Eq. (4) have absolute value greater than one, then the equilibrium point \( \bar{y} \) is a source.

1.3 Theorem [16] Assume that \( p_i \in \mathbb{R}, i = 1, 2, \ldots, k \). Then

\[ \sum_{i=1}^{k} |p_i| < 1, \]

is a sufficient condition for the asymptotically stable of Eq. (39)

\[ y_{n+k} + p_1y_{n+k-1} + \ldots + p_ky_n = 0, \quad n = 0, 1, \ldots. \tag{5} \]

2. Local stability of the equilibrium point

In this section we investigate the local stability character of the solutions of Eq. (1). Eq. (1) has a unique nonzero equilibrium point

\[ \bar{x} = a + \frac{\sum_{i=0}^{k} c_i}{b + d} x, \]

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Let

\[ G = \sum_{i=0}^{k} c_i. \]

Then, we get

\[ \bar{x} = a + \frac{G}{b + d}. \]
Let \( f : (0, \infty)^{k+1} \to (0, \infty) \) be a function defined by
\[
f(u_0, u_1, \ldots, u_k, v) = x_{n+1} = a + \sum_{i=0}^{k} c_i u_i.
\]

Therefore it follows that
\[
\frac{\partial f(u_0, u_1, \ldots, u_k, v)}{\partial v} = -b \sum_{i=0}^{k} c_i u_i\]
and
\[
\frac{\partial f(u_0, u_1, \ldots, u_k, v)}{\partial u_j} = c_j \left( \frac{v + du_i}{b + du_i} \right)^2 = \frac{c_j}{b + du_i}, \quad j = 0, 1, \ldots, k - 1,
\]
and
\[
\frac{\partial f(u_0, u_1, \ldots, u_k, v)}{\partial u_k} = \frac{c_k (v + du_k) - d \sum_{i=0}^{k} c_i}{b + du_k}.
\]
Then we see that
\[
\frac{\partial f(x, \ldots, x_k)}{\partial v} = -bG \left( \frac{v + du_i}{b + du_i} \right)^2 = -P_0,
\]
and
\[
\frac{\partial f(x, \ldots, x_k)}{\partial u_j} = \frac{c_j}{a(b + d)G} = -P_j, \quad j = 0, 1, \ldots, k - 1,
\]
and
\[
\frac{\partial f(x, \ldots, x_k)}{\partial u_k} = \frac{c_k (v + du_k) - d \sum_{i=0}^{k} c_i}{b + du_k} = -P_k.
\]

Then the linearized equation of (1) about \( \bar{x} \) is
\[
z_{n+1} = \sum_{i=0}^{k} p_i z_{n-i}.
\]

2.1 Theorem Assume that
\[
(b - d)G < a(b + d)^2.
\]
Then the equilibrium point of Eq. (1) is locally stable.

Proof It is follows by Theorem (1.3) that, Eq. (7) is locally stable if
\[
\left| c_1 \right| + \ldots + \left| c_k \right| + |c_0| < 1.
\]
That is
\[
c_j (b + d) + b \sum_{i=0}^{k} c_i + c_k (b + d) - d \sum_{i=0}^{k} c_i < (b + d)(a(b + d) + G)
\]
If
\[
c_k (b + d) > d \sum_{i=0}^{k} c_i
\]
this implies that

\[ G(b + d) + G(b - d) < a(b + d)^2 + G(b + d). \]

Thus

\[ (b - d)G < a(b + d)^2. \]

Hence, the proof is completed.

### 3. Periodic solutions

In this section we investigate the periodic character of the positive solutions of Eq. (1).

#### 3.1 Theorem

Eq. (1) has positive prime period two solution if

\[ \ell - \text{even}, k - \text{odd and } (b - d)(ad + \beta - ab - \alpha) > 4d(ab + \alpha), \]

\[ (i) \]

\[ (ii) \ell - \text{odd}, k - \text{even and } (d - b)(ab + \beta - ad - \alpha) > 4b(ad + \alpha). \]

#### Proof

For case (i) assume that there exists a prime period-two solution

..., p, q, p, q, ...

of (1). Let \( x_n = q, x_{n+1} = p \). Since \( \ell - \text{even}, k - \text{odd} \) we have \( x_{n-\ell} = q, x_{n-k} = p \). Thus, from Eq. (1), we get

\[ p = a + \frac{c_0 q + c_1 p + c_2 q + \ldots + c_k p}{bq + dp}, \]

and

\[ q = a + \frac{c_0 p + c_1 q + c_2 p + \ldots + c_k q}{bp + dq}. \]

Let

\[ c_0 + c_2 + \ldots + c_{k-1} = \alpha, \]

and

\[ c_1 + c_3 + \ldots + c_k = \beta. \]

Then

\[ p = a + \frac{\alpha q + \beta p}{bq + \beta p}, \]

and

\[ q = a + \frac{\alpha p + \beta q}{bp + \beta q}. \]

Then

\[ bpq + dp^2 = abq + adp + \alpha q + \beta p, \]

and

\[ bpq + dq^2 = abp + adq + \alpha p + \beta q. \]

Subtracting (10) from (11) gives

\[ d(p^2 - q^2) = (ad + \beta - ab - \alpha)(p - q). \]

Since \( p \neq q \), we have
\[ p + q = \frac{ad + \beta - ab - \alpha}{d}. \]  

(12)

Also, since \( p \) and \( q \) are positive, \( (ad + \beta - ab - \alpha) \) should be positive. Again, adding (10) and (11) yields

\[ 2bpq + d(p^2 + q^2) = (ad + \beta + ab + \alpha)(p + q). \]  

(13)

It follows by (12), (13) and the relation

\[ p^2 + q^2 = (p + q)^2 - 2pq, \quad \forall p, q \in \mathbb{R}, \]

that

\[ pq = \frac{(ab + \alpha)(ad + \beta - ab - \alpha)}{d(b - d)}. \]  

(14)

It is clear now, from Eq. (12) and Eq. (14) that \( p \) and \( q \) are the two distinct roots of the quadratic equation

\[ t^2 - \left( \frac{ad + \beta - ab - \alpha}{d} \right)t + \frac{(ab + \alpha)(ad + \beta - ab - \alpha)}{d(b - d)} = 0, \]

and so

\[ \left( \frac{ad + \beta - ab - \alpha}{d} \right)^2 - \frac{4(ab + \alpha)(ad + \beta - ab - \alpha)}{d(b - d)} > 0, \]

which is equivalent to

\[ (b - d)(ad + \beta - ab - \alpha) > 4d(ab + \alpha). \]

The proof follows by induction. The case where (ii) holds is similar and will be omitted. This completes the proof.

4. Bounded Solution

Our aim in this section we investigate the boundedness of the positive solutions of Eq. (1).

4.1 Theorem Let \( \{x_n\} \) be a solution of Eq. (1). Then the following statements are true:

1) Suppose \( G < d \) and for some \( N \geq 0 \), the initial condition

\[ x_{N-i+1}, \ldots, x_{N-k+1}, \ldots, x_{N-1}, x_N \in \left[ \frac{G}{d}, 1 \right], \]

then

\[ a + \frac{G}{d^2 + db} \leq x_n \leq a + 1, \quad \text{for all } n \geq N. \]  

(15)

2) Suppose \( G > d \) and for some \( N \geq 0 \), the initial condition

\[ x_{N-i+1}, \ldots, x_{N-k+1}, \ldots, x_{N-1}, x_N \in \left[ \frac{G}{d}, 1 \right], \]

then

\[ a + 1 \leq x_n \leq a + \frac{G}{d^2 + db}, \quad \text{for all } n \geq N. \]

Proof First of all, if for some \( N \geq 0 \), \( \frac{G}{d} \leq x_N \leq 1 \) and we have
\[ x_{N+1} = a + \frac{\sum_{i=0}^{k} c_i x_{N-i}}{b x_{N-\ell} + d x_{N-k}} \]

\[ x_{N+1} = a + \frac{G x_{N-i}}{b x_{N-\ell} + d x_{N-k}} \leq a + \frac{G}{d(\frac{\varepsilon}{d})}, \]

then

\[ x_{N+1} \leq a + 1. \tag{16} \]

Similarly, we can show that

\[ x_{N+1} \geq a + \frac{G(\frac{\varepsilon}{d})}{b x_{N-\ell} + d x_{N-k}}, \]

then

\[ x_{N+1} \geq a + \frac{G^2}{d^2 + d b}. \tag{17} \]

From (16) and (17) we deduce for all \( n \geq N \) that the inequality (15) is valid. Hence, the proof of part (i) is completed. Similarly, if \( 1 \leq x_N \leq \frac{G}{d} \), then we can prove part (ii) which is omitted here for convenience. Thus, the proof is now completed.

5. Global Stability of Eq. (1)

Our aim in this section we investigate the global asymptotic stability of Eq. (1).

Remark 5.1 If \( c_k \left(b v + d u_k\right) > d \sum_{i=0}^{k} c_i u_i \),

Then the function \( f \left(u_0, u_1, \ldots, u_k, v\right) \) defined by Eq. (6) is non decreasing in \( v \); non increasing in the rest of arguments.

5.1 Theorem If \( ab = G + ad \) then the equilibrium point \( \bar{x} \) of Eq. (1) is global attractor.

Proof Let \( f : (0, \infty)^{k+1} \rightarrow (0, \infty) \) be a function defined by Eq. (6). Therefore

\[ \frac{\partial f(u_0, u_1, \ldots, u_k, v)}{\partial v} = -b \sum_{i=0}^{k} c_i u_i \frac{1}{b v + d u_k}, \]

\[ \frac{\partial f(u_0, u_1, \ldots, u_k, v)}{\partial u_j} = \frac{c_j}{b v + d u_k}, \quad j = 0, 1, \ldots, k-1, \]

and

\[ \frac{\partial f(u_0, u_1, \ldots, u_k, v)}{\partial u_k} = \frac{c_k (b v + d u_k) - d \sum_{i=0}^{k} c_i u_i}{b v + d u_k^2}. \]

We can see that the function \( f(u_0, u_1, \ldots, u_k, v) \) decreasing in \( v \) and increasing in the rest of arguments.

Suppose that \((m, M)\) is a solution of the system

\[ m = f(m, m, \ldots, m, M) \quad \text{and} \quad M = f(M, M, \ldots, M, m). \]
Then from Eq. (6), we see that

\[
m = a + \frac{\sum_{i=0}^{k} c_i m}{bM + dm}, \quad M = a + \frac{\sum_{i=0}^{k} c_i M}{bm + dM},
\]

\[
m = a + \frac{Gm}{bM + dm}, \quad M = a + \frac{GM}{bm + dM},
\]

\[
abM + aM + Gm, \quad abM + adM + GM,
\]

greater than

\[
(m - M)(d(m + M) - (G + ad - ab)) = 0.
\]

Thus

\[
m = M.
\]

It follows by Theorem (1.1) that \( x \) is a global attractor of Eq. (1) and then the proof is complete.

6. Numerical examples

For confirming the results of this section, we consider numerical examples which represent different types of solution of Eq. (1)

6.1 Example Consider the difference equation

\[
x_{n+1} = 1 + \frac{0.01x_n + 18x_{n-1}}{5x_{n-1} + 0.02x_{n-1}},
\]

Where \( k = 1, \ell = 1, a = 1, b = 5, \alpha = c_0 = 0.01, \beta = c_1 = 18, d = 0.02. \) \( \text{Figure(b1), shows that the equilibrium point of Eq.(1) has locally stable, with initial data} \ x_{-1} = 0.3, x_0 = 6.9. \)

6.2 Example Consider the difference equation

\[
x_{n+1} = 1 + \frac{0.5x_n + 0.5x_{n-1}}{2x_{n-1} + x_{n-1}}
\]
where \( k = 1, \ell = 1, a = 1, b = 2, c_0 = 0.5, c_1 = 0.5, d = 1 \). Figure (b2), shows that the equilibrium point of Eq. (1) has global stability, with initial data \( x_{-2} = 0.5, x_{-1} = 3.2, x_0 = 0.9 \).

![Figure 6.2](image)

### 6.3 Example

Consider the difference equation

\[ x_{n+1} = 0.0625 + \frac{0.00625x_n + 1000x_{n-1}}{0.125x_{n-2} + 0.0625x_{n-1}}. \]

where \( k - odd = 1, \ell - even = 2, a = 0.0625, c_0 = 0.00625, \alpha = c_1 = 1000, b = 0.125, d = 0.0625 \). Figure (b4), shows that Eq. (1) which is periodic with period two. Where the initial data satisfies condition (8) of Theorem (3.1) \( x_{-2} = 1.3, x_{-1} = 0.9, x_0 = 2.1 \). (see Table (3.1))

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Table 6.3
Remark Note that the special cases of Eq. (1) have been studied in [1] when
\(a = A, b = 0, d = y, c_0 = 1, c_1 = 0, i \geq 1\) and in [2] when \(k = 1, \ell = 0, b = 1, d = 0, c_0 = 0, c_1 = 1\) and in [3] when
\(k = 2, \ell = 0, a = 0, b = B, d = D, c_0 = 0, c_1 = 0\) and in [4] when
\(k = 2, \ell = 1, a = 0, c_0 = 0, c_1 = 0, c_2 = 0\) and in [5] when
\(k = 1, \ell = 0, a = 0, c_0 = 0, c_1 = 0, c_2 = 0\) and in [6] when
\(k = 1, \ell = 0, a = 0, c_0 = 0, c_1 = 0, c_2 = 0\) and in [7] when \(k = 1, \ell = 0, a = A, b = 1, d = 0, c_1 = 0, i < k\)

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REFERENCES


[3] R. DeVault, S. W. Schultz, On the dynamics of \(x_{n+1} = \frac{c_{n+1}}{a_{n+1} + b_{n+1}}\), Domm. Appl. Nonlinear Analysis, 12 (2005), 35-40.


[5] E. M. Elabbasy, H. El-Metwally and E. M. Elsayed, On the difference equation \(x_{n+1} = \alpha x_n - \frac{b_n}{c_n x_{n-1}}\), Adv. Difference Equ., pages Art. ID, 10 (2006), 82579.


[9] M. A. El-Moneam, E. M. E. Zayed, Dynamics of the rational difference equation \(x_{n+1} = Ax_n + Bx_{n-k} + Cx_{n-l} + \frac{b_{n-k}x_{n-k} + b_{n-l}x_{n-l}}{d_{n-k}x_{n-k} + d_{n-l}x_{n-l}}\), DCDIS Ser. A: Math. Anal. 21 (2014), 317–331.


[12] S. Kalabusic and M. R. S. Kulenovic, On the recursive sequence \( x_{n+1} = \frac{\alpha x_{n-1} + \beta x_{n-2}}{\gamma x_{n-1} + \delta x_{n-2}} \), J. difference. Equations Appl., 9(8) (2003), 701-720.

[13] G. Karakostas and S. Stevic, On the recursive sequence \( x_{n+1} = A + \frac{f(x_{n}, \ldots, x_{n-k+1})}{x_{n-k}} \), Comm. Appl. Nonlinear Analysis, 11 (2004), 87-100.


