A New Type of Fuzzy Implicative Ideals of a BH-Algebra

Husein Hadi Abbass\textsuperscript{a}, Suad Abdulaali Neamah\textsuperscript{b}

\textsuperscript{a}Department of Mathematics, Faculty of Education for Girls
University of Kufa, Iraq

\textsuperscript{b}Department of Mathematics, Faculty of Education for Girls
University of Kufa, Iraq

Abstract

In this paper, the fuzzy implicative ideal with respect to an element of a BH-algebra is introduced and some related properties are investigated. Some relationships among this notion and other types fuzzy ideals of BH-algebra are given.

Indexing terms/Keywords

BCI-algebra, BH-algebra, implicative ideal of BH-algebra, b-implicative ideal of BH-algebra, fuzzy ideal of BH-algebra.

1. INTRODUCTION


In this paper, we give the concept of the fuzzy implicative ideal A with respect to an element of a BH-algebra with some theorems, propositions and examples.

Definition (1.1): (K. ISEKI [7]) A BCI-algebra is an algebra \((X, *, 0)\) of type \((2, 0)\), where \(X\) is a nonempty set, \(*\) is a binary operation and 0 is a constant, satisfying the following axioms, \(\forall x, y, z \in X:\)

i. \((x * y) * (x * z) * (z * y) = 0,\)

ii. \((x * (x * y)) * y = 0,\)

iii. \(x * x = 0,\)

iv. \(x * y = 0 \text{ and } y * x = 0 \text{ imply } x = y.\)

Definition (1.2): (Y. B. Jun, E. H. Roh and H. S. Kim [8]) A BH-algebra is a nonempty set \(X\) with a constant 0 and a binary operation \(*\) satisfying the following conditions:

i. \(x * x = 0, \forall x \in X.\)

ii. \(x * y = 0 \text{ and } y * x = 0 \text{ imply } x = y, \forall x, y \in X.\)

iii. \(x * 0 = x, \forall x \in X.\)

Theorem (1.1). (Y. B. Jun, E. H. Roh and H. S. Kim [8]) Every BH-algebra satisfying the condition:

\((x * y) * (x * z) * (z * y) = 0, \forall x, y, z \in X\) is a BCI-algebra. We denote this condition by \((b_3).\)
Definition (1.3) : (H. H. Abbass and H. M. A. Saeed [1]) Let \( X \) be a BH-algebra. Then the set \( X_0 = \{ x \in X \mid 0^* x = 0 \} \) is called the BCA-part of \( X \).

Definition (1.4), (J. Meng and XL. X [6]) A BCI-algebra is said to be an implicative if it satisfies the condition:

\( (x^*(y^*)^*)^*(y^*) = y^*(y^*) \), \( \forall x, y \in X \). We generalize the concept of an implicative to a BH-algebra.

Definition (1.5). A BH-algebra is said to be an implicative if it satisfies the condition,

\( (x^*(y^*)^*)^*(y^*) = y^*(y^*) \); \( \forall x, y \in X \).

Example (1.1). Consider the BH-algebra \( X = \{ 0, 1, 2 \} \) with the binary operation \( * \) defined by the following table:

<table>
<thead>
<tr>
<th>*</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

Then \( (X^*, 0) \) is an implicative BH-algebra.

Remark (1.1). (Y. B. Jun, H. S. Kim and M. Kondo [9]) Let \( X \) and \( Y \) be BH-algebras. A mapping \( f : X \rightarrow Y \) is called a homomorphism if \( f(x^y) = f(x)^* f(y) \), \( \forall x, y \in X \). A homomorphism \( f \) is called a monomorphism (resp., epimorphism) if it is injective (resp., surjective). A bijective homomorphism is called an isomorphism. Two BH-algebras \( X \) and \( Y \) are said to be isomorphic, written \( X \cong Y \), if there exists an isomorphism \( f : X \rightarrow Y \). For any homomorphism \( f : X \rightarrow Y \), the set \( \{ x \in X \mid f(x) = 0 \} \) is called the kernel of \( f \), denoted by \( \ker(f) \), and the set \( \{ f(x) : x \in X \} \) is called the image of \( f \), denoted by \( \text{Im}(f) \). Notice that \( f(0) = 0 \), \( \forall \) homomorphism \( f \).

Definition (1.6). (H. H. Abbass and H. M. A. Saeed [1]) A nonempty subset \( I \) of a BH-algebra \( X \) is called an implicative ideal of \( X \) if:

\( i. \) \( 0 \in I \).

\( ii. \) \( (x^*(y^*)^*)^* z \in I \) and \( z \in I \) imply \( x \in I \), \( \forall x, y, z \in X \).

Proposition (1.1). (H. H. Abbass and H. M. A. Saeed [1]) Every implicative ideal of a BH-algebra \( X \) is an ideal of \( X \).

Definition (1.7) : (L. A. Zadeh [8]) Let \( X \) be a non-empty set and \( I \) be the closed interval \([0, 1] \) of the real line (real numbers). A fuzzy set \( A \) in \( X \) (a fuzzy subset of \( X \)) is a function from \( X \) into \([0, 1] \).

Definition (1.8): (L. A. Zadeh [8]) Let \( A \) and \( B \) be two fuzzy sets in \( X \), then:

1. \( A = B \) if and only if \( A(x) = B(x) \), \( \forall x \in X \).
2. \( A \subseteq B \) if and only if \( A(x) \leq B(x) \), \( \forall x \in X \).
3. \( A \subseteq B \) if and only if \( A(x) < B(x) \), \( \forall x \in X \). where \( A \) is called a proper fuzzy subset of \( B \).
4. Through part (2), we can deduce that \( x \in A \) if and only if \( A(x) \geq t \).

Definition (1.9): (H. H. Abbass and H. M. A. Saeed [1]) Let \( A \) and \( B \) be two fuzzy sets in \( X \), then

\( i. \) \( (A \cup B)(x) = \min \{ A(x), B(x) \}, \forall x \in X \)

\( ii. \) \( (A \cap B)(x) = \max \{ A(x), B(x) \}, \forall x \in X \)

\( A \cup B \) and \( A \cap B \) are fuzzy sets in \( X \)

In general, if \( \{ A_\alpha, \alpha \in I \} \) is a family of fuzzy sets in \( X \), then

\( \bigcap_{i=1}^n A_i(x) = \inf \{ A_i(x), i \in I \}, \forall x \in X \) and \( \bigcup_{i=1}^n A_i(x) = \sup \{ A_i(x), i \in I \}, \forall x \in X \). which are also fuzzy sets in \( X \).

Definition (1.10): (M. Ganesh [10]) Let \( A \) be a fuzzy set in \( X \), \( \forall \alpha \in [0, 1] \), the set \( A_\alpha = \{ x \in X, A(x) \geq \alpha \} \) is called a level subset of \( A \). Note that, \( A_0 \) is a subset of \( X \) in the ordinary sense.
Definition (1.11): (E. Mkim and S.S. Ahn [2]) Let X and Y be any two sets, A be any fuzzy set in X and f: X → Y be any function. The set \( f^{-1}(y) = \{ x \in X | f(x) = y \} \), ∀ y ∈ Y. The fuzzy set \( B \) in \( Y \) defined by

\[ B(y) = \begin{cases} \sup_{x \in f^{-1}(y)} A(x) & \text{if } f^{-1}(y) \neq \emptyset \\ \emptyset & \text{otherwise} \end{cases} \]

, \∀ y ∈ Y, is called the image of \( A \) under \( f \) and is denoted by \( f(A) \).

Definition (1.12): (E. Mkim and S.S. Ahn [2]) Let X and Y be any two sets, f: X → Y be any function and B be any fuzzy set in \( f(A) \). The fuzzy set \( A \) in \( X \) defined by

\[ A(x) = B(f(x)) \], \forall x ∈ X \]

is called the preimage of \( B \) under \( f \) and is denoted by \( f^{-1}(B) \).

Definition (1.13): (Q. Zhang, E. H. Roh and Y. B. Jun [11]) A fuzzy subset \( A \) of a BH-algebra \( X \) is said to be a fuzzy ideal if and only if:

i. \( A(0) \supseteq A(x) \), \forall x ∈ X.

ii. \( A(x) \supseteq \min(A(x^*y), A(y)) \), \forall x, y ∈ X.

Proposition (1.2): (H. H. Abbass and H. A. Dahham [3]) Let \( \{A_{\alpha} | \alpha \in \lambda \} \) be a family of fuzzy ideals of a BH-algebra \( X \.

Then \( \bigcap_{\alpha \in \lambda} A_{\alpha} \) is a fuzzy ideal of \( X \).

Proposition (1.3): (H. H. Abbass and H. A. Dahham [3]) Let \( \{A_{\alpha} | \alpha \in \lambda \} \) be a chain of fuzzy ideals of a BH-algebra \( X \.

Then \( \bigcup_{\alpha \in \lambda} A_{\alpha} \) is a fuzzy ideal of \( X \).

Proposition (1.4): (E. Mkim and S.S. Ahn [2]) Let f: (\( X^*, 0 \)) → (\( Y^*, 0' \)) be BH-homomorphism. If \( B \) be a fuzzy ideal of \( Y \), then \( f^{-1}(B) \) is a fuzzy ideal of \( X \).

Proposition (1.5): (E. Mkim and S.S. Ahn [2]) Let f: (\( X^*, 0 \)) → (\( Y^*, 0' \)) be BH-epimorphism. If \( A \) is a fuzzy ideal of \( X \), then \( f(A) \) is a fuzzy ideal of \( Y \).

Definition (1.14): (H. H. Abbass and H. M. A. Saeed [4]) A fuzzy ideal \( A \) of a BH-algebra \( X \) is said to be fuzzy closed if \( A(0^*x) \supseteq A(x) \), \forall x ∈ X.

Definition (1.15): (H. H. Abbass and H. M. A. Saeed [4]) A fuzzy subset \( A \) of a BH-algebra \( X \) is called a fuzzy implicative ideal of \( X \) if it satisfies:

i. \( A(0) \supseteq A(x) \), \forall x ∈ X.

ii. \( A(x) \supseteq \min \{ A(x^*(y^*x)^*z), A(z) \} \), \forall x, y, z ∈ X.

Definition (1.16): (H. H. Abbass and S. A. Neamah [5]) A nonempty subset \( I \) of a BH-algebra \( X \) is called an implicative ideal with respect to an element \( b \) of a BH-algebra (or briefly b-implicative ideal), \( b \in X \). if

i. \( 0 \in I \).

ii. \( ((x^*(y^*x))^*z)^*b \in I \) and \( z \in I \) imply \( x \in I \), \forall x, y, z ∈ X.

Proposition (1.6): (H. H. Abbass and H. M. A. Saeed [4]) In BH-algebra of \( X \), every fuzzy implicative ideal is a fuzzy ideal, but the converse is not true in general.

Theorem (1.2): (L. A. Zadeh [8]) A BCI-algebra \( X \) is an implicative if and only if every fuzzy closed ideal of \( X \) is a fuzzy implicative ideal of \( X \).

Theorem (1.3)(H. H. Abbass and S. A. Neamah [5]) Let \( X \) be a BH-algebra and satisfies the condition:

\( ((x^*)y)^*(x^*z)^*z^*y = 0, \forall x, y, z \in X \ (b_0) \).

Then \( X \) is an implicative if and only if every fuzzy closed ideal of \( X \) is a fuzzy implicative ideal of \( X \).

2. A Fuzzy Implicative Ideal with Respect to an Element of a BH-algebra.

In this section, a new notion of fuzzy implicative ideal of a BH-algebra , namely a fuzzy implicative ideal with respect to an element of a BH-algebra, is introduced and some related properties are investigated and we state and prove some propositions and theorems.

Definition (2.1): Let \( X \) be a BH-algebra and \( b \) be an element of \( x \). A fuzzy subset \( A \) of \( X \) is called a fuzzy implicative ideal with respect to an element \( b \) (or briefly, fuzzy b-implicative ideal) of \( X \) if it satisfies:

i. \( A(0) \supseteq A(x), \forall x \in X \).

ii. \( A(x) \supseteq \min \{ A((x^*(y^*x))^*z)^*b), A(z) \} , \forall x, y, z \in X \).

Example (2.1): Consider the BH-algebra \( X = \{0, 1, 2, 3 \} \) with the following operation table:
The fuzzy subset $A$ of $X$ defined by

$$A(x) = \begin{cases} 1 & ; x = 0, 3 \\ 0.5 & ; x = 1, 2 \end{cases}$$

Then $A$ is a fuzzy 3-implicative ideal of $X$.

**Remark (2.1).** A fuzzy b-implicative ideal of a BH-algebra $X$ may not be a fuzzy implicative ideal of $X$, as in the following example.

**Example (2.2).** The fuzzy 3-implicative ideal $A$ of $X$ in example (2.1) is not a fuzzy implicative ideal of $X$. Since if $x = 1, y = 2, z = 0$, then $A(x) = 0.5 < \min \{A((1*(2*1))*0), A(0)\} = \min \{A(1*2), A(0)\} = \min \{A(0), A(0)\} = A(0) = 1$

**Theorem (2.1).** Let $X$ be a BH-algebra. Then $A$ is a fuzzy implicative ideal of $X$ if and only if $A$ is a fuzzy 0-implicative ideal of $X$.

**Proof:** Let $A$ be a fuzzy implicative ideal of $X$. Then

i. $A(0) \geq A(x), \forall x \in X$. [By definition (1.15)(i)]

ii. Let $x, y, z \in X$. Then we have $A(x) \geq \min \{A((x*y)*y)*z), A(z)\}$ [By definition (1.15)(ii)]

$\Rightarrow \min \{A((x*(y*x)))*z), A(z)\} = \min \{A((x*(y*x)))*z), A(z)\}$ [Since $X$ is a BH-algebra; $x*0=x, \forall x \in X$]

$\Rightarrow A(x) \geq \min \{A((x*(y*x)))*z), A(z)\}$. Therefore, $A$ is a fuzzy 0-implicative ideal of $X$.

**Conversely,** Let $A$ be a fuzzy 0-implicative ideal of $X$. Then

i. $A(0) \geq A(x), \forall x \in X$. [By definition (2.1)(i)]

ii. Let $x, y, z \in X$. Then $A(x) \geq \min \{A((x*(y*x)))*z), A(z)\}$ [Since $A$ is a fuzzy 0-implicative ideal of $X$. By definition (2.1)(ii)]

$\Rightarrow \min \{A((x*(y*x)))*z), A(z)\} = \min \{A((x*(y*x)))*z), A(z)\}$ [Since $X$ is a BH-algebra; $x*0=x, \forall x \in X$]

$\Rightarrow A(x) \geq \min \{A((x*(y*x)))*z), A(z)\}$. Therefore, $A$ is a fuzzy implicative ideal of $X$.

**Proposition (2.1).** Let $X$ be a BH-algebra, $b \in X$ and $A$ be a fuzzy b-implicative ideal of $X$, such that $A(b) = A(0)$. Then $A$ is a fuzzy ideal of $X$.

**Proof:** Let $A$ be a fuzzy b-implicative ideal of $X$. To prove $A$ is a fuzzy ideal of $X$.

i. $A(0) \geq A(x), \forall x \in X$. [By definition (2.1)(i)]

ii. Let $x \in X$. Then $A(x*b)=A((x*0)*b)$ [Since $X$ is BH-algebra; $x*0=x$]

$=A((x*(x*x))*b)$. [Since $X$ is BH-algebra; $x*x=0, \forall x \in X$]

$=A((x*(x*x))*b)$. [Since $X$ is BH-algebra; $x*0=x, \forall x \in X$]

Now, we have $\min \{A((x*(x*x))*0)*b), A(0)\}=A(x*b)$

$\Rightarrow A(x) \geq \min \{A((x*(x*x))*0)*b), A(0)\}$ [By definition (2.1)(ii)]

$\Rightarrow A(x) \geq \min \{A(x*b), A(0)\}$. [Since $A(b) = A(0)$]. Therefore, $A$ is a fuzzy ideal of $X$.

**Remark (2.2).** The following example shows that converse of proposition (2.1) is not correct, $\forall b \in X$. 

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
Example (2.3). Consider the BH-algebra $X = \{0, 1, 2\}$ with the binary operation $\ast$ defined by the following table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

Then the fuzzy set $A$ which defined by

\[
A(x) = \begin{cases} 
1, & x = 0 \\
0.5, & x = 1 \\
0, & x = 2 
\end{cases}
\]

is a fuzzy ideal of $X$.

But $A$ is not a fuzzy 0-implicative ideal of $X$. Since if $x = 2$, $y = 0$, $z = 0$, then $A(2) = 0.5 < \min\{A((2 \ast (0 \ast 2)) \ast 0), A(0)\} = \min\{A(0), A(0)\} = 1$.

And $A$ is not a fuzzy 1-implicative ideal of $X$. Since if $x = 1$, $y = 0$, $z = 0$, then $A(2) = 0.5 < \min\{A((1 \ast (0 \ast 1)) \ast 0) \ast 1), A(0)\} = \min\{A(0), A(0)\} = A(0) = 1$.

And $A$ is not a fuzzy 2-implicative ideal of $X$. Since if $x = 2$, $y = 2$, $z = 0$, then $A(2) = 0.5 < \min\{A((2 \ast (2 \ast 2)) \ast 0) \ast 2), A(0)\} = \min\{A(0), A(0)\} = A(0) = 1$.

Therefore, $A$ is not a fuzzy $b$-implicative ideal.

Proposition (2.2). Let $X$ be a BH-algebra. $A$ be a fuzzy implicative ideal of $X$, $b \in X$ such that $A(b) = A(0)$. Then $A$ is a fuzzy $b$-implicative ideal of $X$.

Proof: Let $A$ be a fuzzy implicative ideal of $X$. Then

i. $A(0) \geq A(x), \ \forall \ x \in X$ [By definition (1.13)(i)]

ii. Let $x, y, z \in X$. Then $A(x \ast (y \ast x)) \geq \min\{A((x \ast (y \ast x)) \ast z), A(z)\}$ [Since $A$ is a fuzzy ideal of $X$. By definition (1.13)(ii)]

\[
= \min\{A((x \ast (y \ast x)) \ast z) \ast b), A(z)\} \quad [Since A is a fuzzy ideal of X.]
\]

$\Rightarrow A$ is a fuzzy $b$-implicative ideal of $X$. Therefore, $A$ is fuzzy $b$-implicative ideal of $X$.

Remark (2.3). The following example shows that the converse of proposition (2.2) is not correct in general.

Example (2.4). Consider the fuzzy 3-implicative ideal $A$ of the BH-algebra $X$ in example (2.1), $A$ is not a fuzzy implicative ideal of $X$. Since if $x = 1$, $y = 2$, $z = 0$, then $A(1) = 0.5 < \min\{A((1 \ast (2 \ast 1)) \ast 0), A(0)\} = \min\{A(1), A(0)\} = A(0) = 1$.

Theorem (2.2). Let $X$ be a BH-algebra. Then a fuzzy ideal $A$ of $X$ satisfying the condition:

\[
\forall \ x, y \in X; \ A(x) \geq A(x \ast (y \ast x)) \quad (b_1)
\]

is a fuzzy $b$-implicative ideal of $X$, where $b \in X$ and $A(b) = A(0)$.

Proof: Let $A$ be a fuzzy ideal of $X$. Then, we have

i. $A(0) \geq A(x), \ \forall \ x \in X$. [By definition (1.13)(i)]

ii. Let $x, y, z \in X$. Then, we have $A(x \ast (y \ast x)) \geq \min\{A((x \ast (y \ast x)) \ast z), A(z)\}$ [Since $A$ is a fuzzy ideal of $X$. By definition (1.13)(ii)]

\[
\geq \min\{\min\{A(((x \ast (y \ast x)) \ast z) \ast b), A(b)\}, A(z)\} \quad [Since A is a fuzzy ideal of X.]
\]

\[
= \min\{A(((x \ast (y \ast x)) \ast z) \ast b), A(z)\} \quad [Since A is a fuzzy ideal of X.]
\]

$\Rightarrow A$ is a fuzzy $b$-implicative ideal of $X$. $A(b) = A(0)$.

Theorem (2.3). If $X$ is a BH-algebra of $X$ satisfies the condition: $\forall \ x, y \in X; \ x = x \ast (y \ast x) \quad (b_2)$, then every fuzzy ideal of $X$ is a fuzzy $b$-implicative ideal of $X$, where $b \in X$ and $A(b) = A(0)$.

Proof: Let $A$ be a fuzzy ideal of $X$. Then, we have

i. $A(0) \geq A(x), \ \forall \ x \in X$. [By definition (1.13)(i)]

ii. Let $x, y, z \in X$. Then $A(x \ast (y \ast x)) \geq \min\{A((x \ast (y \ast x)) \ast z), A(z)\}$ [Since $A$ is a fuzzy ideal of $X$. By definition (1.13)(ii)]
Now, $A(x) = A(x*(y*x))$ [By $(b_2)$]

$\Rightarrow A(x) \geq \min \{A((x*(y*x))z), A(z)\}$ [By definition (1.13)(ii)]

$\Rightarrow A$ is a fuzzy implicative ideal of $X$. [By definition (1.15)]

Therefore, $A$ is a fuzzy b-implicative ideal of $X$. [By proposition (2.2)]. ■

**Theorem (2.4).** Let $X = X_+$ be an implicative BH-algebra satisfies $((x*y) * (x*z))*z*y = 0$, $\forall x, y, z \in X$. (b3),

Then $A$ is a fuzzy ideal of $X$ if and only if it is a fuzzy 0-implicative ideal of $X$.

**Proof:** Let $A$ be a fuzzy ideal of $X$ and $x \in X$. Then $A(0*x) = A(0)$.

[Since $X = X_+$. By definition (1.3)]

$\Rightarrow A(0*x) \geq A(x)$. [A(0) \geq A(x), \forall x \in X. By definition (1.13)(i)]

$\Rightarrow A$ is a fuzzy closed ideal of $X$. [By definition (1.14)]

Since $X$ is BH-algebra satisfies (b3), then $X$ is BCI-algebra. [By theorem (1.1)]

$\Rightarrow A$ is a fuzzy implicative ideal of $X$. [By theorem (1.2)]

$\Rightarrow A$ is a fuzzy 0-implicative ideal of $X$. [By theorem (2.2)]. ■

**Conversely,** Let $A$ be a fuzzy 0-implicative ideal of $X$.

$\Rightarrow A$ is a fuzzy implicative ideal of $X$. [By proposition (2.1)]

$\Rightarrow A$ is a fuzzy ideal of $X$. [By definition (1.6)]. ■

**Theorem (2.5).** Let $X = X_+$ be an implicative BH-algebra satisfies (b3). Then a fuzzy ideal $A$ of $X$ is a fuzzy b-implicative ideal of $X$, where $b \in X$ and $A(b) = A(0)$.

**Proof:** Let $A$ be a fuzzy b-implicative ideal of $X$. To prove $X_A$ is a b-implicative ideal of $X$.

i. $A(x) = A(0)$. If $x = 0$, then $0 \in X_A$

ii. Let $x, y, z, b \in X$ such that $((x*(y*x))z)b \in X_A$ and $z \in X_A$

$\Rightarrow A(((x*(y*x))z)b) = A(0)$ and $A(z) = A(0)$

[by definition of b-implicative ideal of $X$, we have]

$A(x) \geq \min \{A(((x*(y*x))z)b), A(z)\} = \min \{A(0), A(0)\} = A(0)$

$\Rightarrow A(x) \geq A(0)$. But $A(0) \geq A(x)$. [Since $A$ is a fuzzy b-implicative ideal of $X$]

$\Rightarrow A(x) = A(0)$

$\Rightarrow x \in X_A$. Therefore, $X_A$ is a b-implicative ideal of $X$.

**Proposition (2.3).** Let $\{A_\alpha | \alpha \in \Lambda\}$ be a family of fuzzy b-implicative ideals of a BH-algebra $X$ and $b \in X$. Then $\bigcap_{\alpha \in \Lambda} A_\alpha$ is a fuzzy b-implicative ideal of $X$.

**Proof:** Let $\{A_\alpha | \alpha \in \Lambda\}$ be a family of fuzzy b-implicative of $X$ and $b \in X$. 

i. Let \( x \in X \). Then \( \bigcap \{A_\alpha(0) \geq \inf \{A_\alpha(x) | \alpha \epsilon \lambda \} \} = \inf \{A_{\alpha}(0) | \alpha \epsilon \lambda \} = \bigcup A_\alpha(x) \) \[ \text{[Since } A_\alpha \text{ is a fuzzy b-implicative ideal of } X, \forall \alpha \epsilon \lambda. \text{ By definition (2.1)(i)]. Therefore, } \bigcap A_\alpha(0) \geq \bigcup A_\alpha(x). \]

ii. Let \( x, y, z \in X \) and \( b \in X \). Then, we have \( \bigcap A_\alpha(x) = \inf \{A_\alpha(x) | \alpha \epsilon \lambda \} \geq \inf \{\min\{A_\alpha((((x^*(y^*)x)^*)z)^*)b), A_\alpha(z) | \alpha \epsilon \lambda\}\} \) \[ \text{[Since } A_\alpha \text{ is a fuzzy b-implicative ideal of } X, \forall \alpha \epsilon \lambda. \text{ By definition (2.1)(ii)]} \]

\[
\begin{align*}
\Rightarrow \bigcap A_\alpha(x) &\geq \min\{\bigcap A_\alpha((((x^*(y^*)x)^*)z)^*)b) | \alpha \epsilon \lambda\}, \bigcup A_\alpha(z) | \alpha \epsilon \lambda\} \\
\text{Therefore, } \bigcap A_\alpha(x) \text{ is a fuzzy b-implicative ideal of } X. &\blacksquare
\end{align*}
\]

**Proposition (2.4).** Let \( \{A_\alpha | \alpha \epsilon \lambda\} \) be a chain of fuzzy b-implicative ideals of a BH-algebra \( X \) and \( b \in X \). Then \( \bigcup A_\alpha \) is a fuzzy b-implicative ideal of \( X \).

**Proof:** Let \( \{A_\alpha | \alpha \epsilon \lambda\} \) be a chain of fuzzy b-implicative ideals of \( X \) and \( b \in X \).

i. Let \( x \in X \). Then \( \bigcup A_\alpha(0) = \sup\{A_\alpha(0) | \alpha \epsilon \lambda\} \geq \sup\{A_\alpha(x) | \alpha \epsilon \lambda\} = \bigcup A_\alpha(x) \) \[ \text{[Since } A_\alpha \text{ is a fuzzy b-implicative ideal of } X, \forall \alpha \epsilon \lambda. \text{ By definition (2.1)(i)]. Therefore, } \bigcup A_\alpha(0) \geq \bigcup A_\alpha(x). \]

ii. Let \( x, y, z \in X \) and \( b \in X \). Then, we have \( \bigcup A_\alpha(x) = \sup\{A_\alpha(x) | \alpha \epsilon \lambda\} \geq \sup\{\min\{A_\alpha(((x^*(y^*)x)^*)z)^*)b), A_\alpha(z) | \alpha \epsilon \lambda\}\} \) \[ \text{[Since } A_\alpha \text{ is a fuzzy b-implicative ideal of } X, \forall \alpha \epsilon \lambda. \text{ By definition (2.1)(ii)]} \]

\[
\begin{align*}
\Rightarrow \bigcup A_\alpha(x) &\geq \min\{\bigcup A_\alpha(((x^*(y^*)x)^*)z)^*)b) | \alpha \epsilon \lambda\}, \bigcup A_\alpha(z) | \alpha \epsilon \lambda\} \\
\text{Therefore, } \bigcup A_\alpha(x) \text{ is a fuzzy b-implicative ideal of } X. &\blacksquare
\end{align*}
\]

**Proposition (2.4).** Let \( f: (X^*, 0) \rightarrow (Y^*, 0') \) be a BH-epimorphism. If \( A \) is a fuzzy b-implicative ideal of \( X \), then \( f(A) \) is a fuzzy \( f(b) \)-implicative ideal of \( Y \).

**Proof:** Let \( A \) be a fuzzy b-implicative ideal of \( X \).

i. Let \( y \in Y \). Then, there exists \( x \in X \), \( f(A)(0') = \sup\{A(x_1) | x_1 \epsilon f^{-1}(0')\} \)

\( A(0') = \sup\{A(x) : x \epsilon X \} \geq \sup\{A(x) : x = f^{-1}(y)\} = f(A)(y) \) \[ \text{[Since } A \text{ is a fuzzy b-implicative ideal of } X. \text{ By definition (2.1)(i)]} \]

\( \Rightarrow f(A)(0') \geq f(A)(y) \forall y \epsilon Y. \)

ii. Let \( y_1, y_2, y_3, t \epsilon Y \). Then there exist \( f(x_1) = y_1, f(x_2) = y_2, f(z) = y_3 \) and \( f(b) = t \) such that \( x_1, x_2, z, b \epsilon X \)

\( \Rightarrow f(A)(y_1) = \sup\{A(x_1) | x \epsilon f^{-1}(y_1)\} \)
\[ \sup \{ A \mid \{(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}) \} \in f^{-1}(\{(y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}) \}) \} \]

[Since \( A \) is a fuzzy b-implicative ideal of \( X \). By definition (2.1)(ii)]

\[ \min \{ \sup \{ A \mid \{(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}) \} \in f^{-1}(\{(y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}) \}) \} \}
\]

\[ \min \{ \min \{ A \mid \{(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}) \} \in f^{-1}(\{(y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}) \}) \} \}
\]

[Since \( f \) is an epimorphism. By remark (1.1)]

\[ \Rightarrow f(A)(y) \geq \min \{ (f(A)(y_1), y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}) \}, (f(A)(y)) \}
\]

Therefore, \( f(A) \) is a fuzzy b-implicative ideal or bi-implicative ideal of \( Y \).

**Proposition (2.5)** Let \( f : (X, ^* , 0) \rightarrow (Y, ^* , 0) \) be a BH-homomorphism. If \( B \) is a fuzzy b-implicative ideal of \( Y \), then \( f^{-1}(B) \) is a fuzzy b-implicative ideal of \( X \).

**Proof:** Let \( B \) be a fuzzy b-implicative ideal of \( Y \). To prove \( f^{-1}(B) \) is a fuzzy b-implicative ideal of \( X \).

i. Let \( x \in X \). Then \( (f^{-1}(B))(x) = B(f(x)) = B(0) \geq B(f(x)) = (f^{-1}(B))(x) \) \[Since \( B \) is a fuzzy b-implicative ideal of \( Y \). By definition (2.1)(i)]

\[ \Rightarrow (f^{-1}(B))(x) \geq (f^{-1}(B))(x), \forall x \in X. \]

ii. Let \( x, y, z \in X \). Then

\[ (f^{-1}(B))(x) = B(f(x)) \] [By definition (1.12)]

\[ \geq \min \{ B[(1, (x', y', z')] \} \] [Since \( B \) is a fuzzy b-implicative ideal of \( Y \). By definition (2.1)(i)]

\[ \min \{ B(1, (x', y', z')) \} \] [Since \( f \) is a homomorphism.]

\[ \min \{ (f^{-1}(B))(1, (x', y', z')) \} \]

[Since \( f^{-1}(B)(1, (x', y', z')) \)] [By definition (1.12)]

\[ \Rightarrow (f^{-1}(B))(x) \geq \min \{ (f^{-1}(B))(1, (x', y', z')) \}, (f^{-1}(B))(z) \} \]

Therefore, \( f^{-1}(B) \) is a fuzzy b-implicative ideal of \( X \).

**Proposition (2.6)**: Let \( X \) be a BH-algebra and \( A \) be a fuzzy subset of \( X \). Then \( A \) is a fuzzy b-implicative ideal of \( X \) if and only if \( f(A)(x) = A(x) + 1 - A(0) \) is a fuzzy b-implicative ideal of \( X \) where \( b \in X \).

**Proof:** Let \( A \) be a fuzzy b-implicative ideal of \( X \). Then

i. \( A^0(0) = A(0) + 1 - A(0) \)

\[ \Rightarrow A^0(0) = 1. \text{ Then } A^0(0) \geq A^0(x), \forall x \in X. \]

ii. Let \( x, y, z \in X \) and \( b \in X \). Then \( A^0(x) = A(x) + 1 - A(0) \geq \min \{ A(((x', y', z')) + b), A(z) + 1 - A(0) \} \) [Since \( A \) is a fuzzy b-implicative ideal of \( X \). By definition (2.1)(i)]

\[ \Rightarrow \min \{ A(((x', y', z')) + b), A(z) + 1 - A(0) \} \geq \min \{ A^0(((x', y', z')) + b), A^0(z) \} \]

\[ \Rightarrow A^0(x) \geq \min \{ A^0(((x', y', z')) + b), A^0(z) \} \]

\[ \Rightarrow A^0 \text{ is a fuzzy b-implicative ideal of } X. \]

**Conversely**, Let \( A^0 \) be a fuzzy b-implicative ideal of \( X \).

i. Let \( x \in X \). Then we have \( A(0) = A^0(0) + 1 - A(0) \geq A^0(0) + 1 - A(0) = A(0) \) [Since \( A^0 \) is a fuzzy b-implicative ideal of \( X \). By definition (2.1)(i)]

\[ \Rightarrow A(0) \geq A(x), \forall x \in X. \]

ii. Let \( x, y, z \in X \) and \( b \in X \). Then \( A^0(x) = A^0(x) + 1 - A(0) \geq \min \{ A^0(((x', y', z')) + b), A^0(z) + 1 - A(0) \} \) [Since \( A^0 \) is a fuzzy b-implicative ideal of \( X \). By definition (2.1)(ii)]

\[ \Rightarrow \min \{ A^0(((x', y', z')) + b) + 1 - A(0), A^0(z) + 1 - A(0) \} \geq \min \{ A^0(((x', y', z')) + b), A^0(z) \} \]

\[ \Rightarrow A(x) \geq \min \{ A((x', y', z') + b), A(z) \}. \text{ Then } A \text{ is a fuzzy b-implicative ideal of } X. \]

**Proposition (2.7)**. Let \( X \) be a BH-algebra and let \( w, b \in X \). If \( A \) is a fuzzy b-implicative ideal of \( X \), then \( \uparrow A(w) \) is a fuzzy b-implicative ideal of \( X \).
Proof: Let A be a fuzzy b-implicative ideal of X. Then
i. A(0) ≥ A(x), ∀ x ∈ X.
   [Since A is a fuzzy b-implicative ideal of X. By definition (2.1)(i)]
   ⇒ A(0) ≥ A(w). Then 0 ∈ A(w)
ii. Let x, y, z ∈ X such that ((x*(y*x)))*z)*b ∈ A(w) and z ∈ A(w)
   ⇒ A(w) ≤ A(((x*(y*x)))*z)*b) and A(w) ≤ A(z)
   ⇒ A(w) ≤ min{A(((x*(y*x)))*z)*b) , A(z)}
But A(x) ≥ min{A(((x*(y*x)))*z)*b) , A(z)}  [Since A is a fuzzy b-implicative ideal of X. By definition (2.1)(ii)]
   ⇒ A(w) ≤ A(x)
   ⇒ x ∈ A(w). Therefore, A(w) is a b-implicative ideal of X. ■

Theorem (2.7). Let X be a BH-algebra, A be a fuzzy ideal of X A(b) = A(0). Then A is a fuzzy b-implicative ideal of X if and only if A(α) is a b-implicative ideal of X, ∀ α ∈ [0, A(0)].
Proof: Let A be a fuzzy b-implicative ideal of X. To prove A(α) is a b-implicative ideal of X.
   i. Let x ∈ A(α). Then A(x) ≥ α [By definition (2.3.1)(ii) of A(α)]
   ∴ A(0) ≥ α. Then 0 ∈ A(α).
ii. Let x, y, z, b ∈ X such that ((x*(y*x)))*z)*b ∈ A(α) and z ∈ A(α).
   ⇒ A(((x*(y*x)))*z)*b) ≥ α and A(z) ≥ α [By definition (2.3.1)(ii) of A(α)]
   min{ A(((x*(y*x)))*z)*b) , A(z) } ≥ α
But A(x) ≥ min{ A(((x*(y*x)))*z)*b) , A(z) }  [Since A is a fuzzy b-implicative ideal of X. By definition (2.1)(ii)]
   ⇒ A(x) ≥ α = x ∈ A(α) [By definition (2.3.1)(ii) of A(α)]
Therefore, A(α) is a b-implicative ideal of X, ∀ b ∈ A(α).
Conversely, To prove A is a fuzzy b-implicative ideal of X.
   i. 0 ∈ A(α) [By definition (2.3.1)(i)]. Then A(0) ≥ α = A(x), ∀ x ∈ X.
   ii. Let x, y, z ∈ X such that α = min{ A(((x*(y*x)))*z)*b) , A(z)}
   ⇒ A(((x*(y*x)))*z)*b) ≥ α and A(z) ≥ α
   ⇒ ((x*(y*x)))*z)*b ∈ A(α) and z ∈ A(α).
   ⇒ x ∈ A(α) [Since A(α) is a b-implicative ideal of X. By definition (1.16)(ii)]
   ⇒ A(x) ≥ α
   ⇒ A(x) ≥ min{ A(((x*(y*x)))*z)*b) , A(z) }. Therefore, A is a fuzzy b-implicative ideal of X, A(b) = A(0). ■

References


