QUALITATIVE BEHAVIOUR FOR SECOND ORDER NONLINEAR DELAY DIFFERENTIAL EQUATIONS OF NEUTRAL TYPE

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Abstract

In this work, asymptotic and oscillatory behaviour of the solutions of a class of nonlinear second-order neutral delay differential equations of the form

\[ (E_1) \quad (r(t)(x(t) + p(t)x(t-\tau)))' + q(t)H(x(t-\sigma)) = f(t) \]

and

\[ (E_2) \quad (r(t)(x(t) + p(t)x(t-\tau)))' + q(t)H(x(t-\sigma)) = 0 \]

are studied under various ranges of \( p(t) \). Sufficient conditions are obtained for existence of bounded positive solutions of \( (E_1) \).

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1. Introduction

Recently, an increasing interest in obtaining sufficient conditions for oscillatory or non-oscillatory behavior of different classes of differential and functional differential equations has been manifested. In particular, investigation of neutral differential equations is important since they are encountered in many applications in science and technology and are used, for instance, to describe distributed networks with lossless transmission lines, in the study of vibrating masses attached to an elastic bar, as well as in some variational problems.

In [2], Baculikova et. al. studied the oscillation of the second order neutral differential equations of the form

\[ (E_3) \quad (r(t)(x(t) + p(t)x(\tau(t))))' + q(t)H(x(\sigma(t))) = 0, \]

and obtained results are based on the new comparison theorems. In [3], Dzurina also considered \( (E_3) \) and established some sufficient conditions for oscillation by using comparison theorems. They all are studied \( (E_1) \), under the conditions

\[ \int_0^\infty \frac{dt}{r(t)} = \infty \quad \text{and} \quad 0 \leq p(t) < \infty \quad \text{only}. \]

In [10], Santra has consider first-order neutral delay differential equations of the form

\[ (E_4) \quad (x(t) + p(t)x(t-\tau))' + q(t)H(x(t-\sigma)) = f(t) \]

and

\[ (E_5) \quad (x(t) + p(t)x(t-\tau))' + q(t)H(x(t-\sigma)) = 0 \]

and studied oscillatory behaviour of the solutions of \( (E_4) \) and \( (E_5) \), under various ranges of \( p(t) \). Also, sufficient conditions are obtained for existence of bounded positive solutions of \( (E_4) \).
Motivation by the work of [10], an attempt is made here to establish sufficient conditions for oscillation of nonlinear neutral forced delay differential equation of the form

\[(r(t)(x(t) + p(t)x(t - \tau)))' + q(t)H(x(t - \sigma)) = f(t),\]  \hfill (1.1)

where

\[\tau > 0, \sigma \geq 0, p \in C([0, \infty), \mathbb{R}), q, r \in C([0, \infty], [0, \infty]), f \in C([0, \infty), \mathbb{R})\]

and \(H\) satisfied

\[H \in C(\mathbb{R}, \mathbb{R}) \text{ with } uH(u) > 0 \text{ for } u \neq 0,\]

under the key assumptions

\[(A_1) \int_0^\infty \frac{dt}{r(t)} = \infty \quad \text{and} \quad (A_2) \int_0^\infty \frac{dt}{r(t)} < \infty,\]

and its associated homogenous equation

\[(r(t)(x(t) + p(t)x(t - \tau)))' + q(t)H(x(t - \sigma)) = 0,\]  \hfill (1.2)

is also considered.

The objective of this work is to establish sufficient conditions for oscillation of all solutions of (1.1)/(1.2) and to study existence of bounded positive solutions of (1.1), under various ranges of \(p(t)\). Unlike the work in [2] and [3] an attempt is made here to establish sufficient conditions under which every solution of (1.1) and (1.2) oscillates. Of course, the impact of forcing term is considered. Keeping in view of the influence of forcing function, this work is separated for forced and unforced equations.

By a solution of (1.1)/(1.2) we understand a function \(x \in C([-\rho, \infty), \mathbb{R})\) such that \(x(t) + p(t)x(t - \tau)\) and \((r(t)(x(t) + p(t)x(t - \tau)))'\) is once continuously differentiable and (1.1) or (1.2) is satisfied for \(t \geq 0\), where \(\rho = \max\{\tau, \sigma\} \text{ and } \sup\{|x(t)|: t \geq t_0\} > 0 \text{ for every } t_0 \geq 0\). A solution of (1.1)/(1.2) is said to be oscillatory if it has arbitrarily large zeros. Otherwise, it is called nonoscillatory.

2. Oscillation properties of Eq. (1.1)

In this section, sufficient conditions are obtained for oscillation of solutions of the equation (1.1). In the sequel, we use the following assumptions:

\[(A_1) \text{ There exists } F \in C(\mathbb{R}, \mathbb{R}) \text{ such that } F(t) \text{ changes sign with } -\infty < \liminf_{t \to \infty} F(t) < 0 < \limsup_{t \to \infty} F(t) < \infty \text{ and } f(t) = (r(t)F'(t))^2; }\]

\[(A_2) \text{ there exists } \lambda > 0 \text{ such that } H(u) + H(v) \geq \lambda H(u + v), \text{ for } u, v > 0; \]

\[(A_3) H(uv) \leq H(u)H(v), \text{ for } u, v \in \mathbb{R}^+; \]

\[(A_4) H(-u) = -H(u), \text{ for } u, v \in \mathbb{R}^+; \]

\[(A_5) F^+(t) = \max\{F(t), 0\} \text{ and } F^-(t) = \max\{-F(t), 0\},\]

**Remark 2.1** We may note that if \(x(t)\) is a solution of (1.1)/(1.2), then \(y(t) = -x(t)\) is also a solution of (1.1)/(1.2) provided that \(H\) satisfies \((A_6)\).

**Theorem 2.2** Let \(0 \leq p(t) \leq p < \infty, t \in \mathbb{R}^+.\) Assume that \((A_1)\) and \( (A_2) - (A_5) \) hold. Furthermore assume that

\[(A_6) \int_T^\infty Q(t)H(F^+(t - \sigma))dt = \infty, \quad T > 0 \quad \text{and}\]

\[\int_T^\infty Q(t)H(F^-(t - \sigma))dt = \infty, \quad T > 0 \quad \text{as well.}\]
\[(A_2) \int_0^\infty Q(t)H(F^-(t-\sigma))\,dt = \infty, \ T > 0 \text{ holds, where } Q(t) = \min\{q(t), q(t-\tau)\}, t \geq \tau. \text{ Then every solution of (1.1) oscillates.}\]

**Proof.** Suppose for contrary that \( x(t) \) is a nonoscillatory solution of equation (1.1). So there exists \( t_0 > 0 \) such that \( x(t) > 0 \) or \( x(t) < 0 \). Assume that \( x(t) > 0, \ x(t-\tau) > 0 \) and \( x(t-\sigma) > 0 \) for \( t \geq t_0 \). Setting

\[
z(t) = x(t) + p(t)x(t-\tau), \ t \geq t_0
\]

and

\[
w(t) = z(t) - F(t),
\]

due to \( (A_1) \), it follows from (1.1) that

\[
r(t)w'(t) = -q(t)H(x(t-\sigma)) \leq 0
\]

for \( t \geq t_1 > t_0 + \sigma \). Consequently, \( r(t)w'(t) \) is nonincreasing and \( w'(t), w(t) \) are of constant sign on \([t_2, \infty), t_2 > t_1\). Since \( z(t) > 0 \), then \( w(t) < 0 \) for \( t \geq t_2 \) implies that \( F(t) > 0 \) for \( t \geq t_2 \), which is absurd. Hence, \( w(t) > 0 \) for \( t \geq t_2 \).

In what follows, we consider the cases \( r(t)w'(t) < 0 \) or \( > 0 \) for \( t \geq t_2 \). Let the former hold for \( t \geq t_2 \). So, there exist \( C > 0 \) and \( t_3 > t_2 \) such that \( r(t)w'(t) \leq -C \) for \( t \geq t_3 \). Integrating the relation \( w'(t) \leq -\frac{C}{r(t)}, t \geq t_3 \) from \( t_3 \) to \( t(t > t_3) \), we obtain

\[w(t) - w(t_3) \leq -C \int_{t_3}^t \frac{ds}{r(s)},\]

that is,

\[w(t) \leq w(t_3) - C \left[ \int_{t_3}^t \frac{ds}{r(s)} \right],\]

\[\rightarrow -\infty \text{ as } t \rightarrow \infty,\]

due to \( (A_1) \), a contradiction to the fact that \( w(t) > 0 \) for \( t \geq t_2 \). Hence, \( r(t)w'(t) > 0 \) for \( t \geq t_2 \).

Ultimately, \( z(t) \geq F(t) \) and hence \( \max\{0, F(t)\} = F^+(t) \) for \( t \geq t_2 \). Due to (2.1) and (2.2), (2.3) becomes

\[0 = (r(t)w'(t))' + q(t)H(x(t-\sigma)) + H(p)(r(t-\tau)w'(t-\tau))' + q(t-\tau)H(x(t-\tau-\sigma))\]

for \( t \geq t_2 \) and because of \( (A_3) \), \( (A_5) \) and \( z(t) \leq x(t) + px(t-\tau) \) we find that

\[0 \geq (r(t)w'(t))' + H(p)(r(t-\tau)w'(t-\tau))' + Q(t)[H(x(t-\sigma)) + H(px(t-\tau-\sigma))]\]

\[\geq (r(t)w'(t))' + H(p)(r(t-\tau)w'(t-\tau))' + \lambda Q(t)H(z(t-\sigma)),\]

(2.4)

for \( t \geq t_3 > t_2 + \sigma \). Integrating (2.4) from \( t_3 \) to \( +\infty \), we get

\[
\lambda \left[ \int_{t_3}^\infty Q(t)H(z(t-\sigma))\,dt \right] \leq -[r(t)w'(t) + H(p)(r(t-\tau)w'(t-\tau))]_0^\infty.
\]

Since \( \lim_{t \to \infty} r(t)w'(t) \) exists, then the above inequality becomes
that is,

\[ \lambda \left[ \int_{3}^{\infty} Q(t)H(z(t-\sigma))dt \right] < \infty, \]

which contradicts \((A_8)\).

If \(x(t) < 0\) for \(t \geq t_0\), then we set \(y(t) = -x(t)\) for \(t \geq t_0\) in (1.2) and we obtain that

\[ (r(t)(y(t) + p(t)y(t-\tau)))' + q(t)H(y(t-\sigma)) = \tilde{f}(t), \]

where \(\tilde{f}(t) = -f(t)\), due to \((A_6)\). Let \(\tilde{F}(t) = -F(t)\). Then

\[ -\infty < \liminf_{t \to \infty} \tilde{F}(t) < 0 < \limsup_{t \to \infty} \tilde{F}(t) < \infty \]

and \((r(t)\tilde{F}(t))' = -f(t) = \tilde{f}(t)\) hold. Further \(\tilde{F}^+(t) = F^-(t)\) and \(\tilde{F}^-(t) = F^+(t)\). Proceeding as above for (2.5), we can find a contradiction to \((A_6)\). Thus, the proof of the theorem is complete.

**Theorem 2.3** Let \(-1 < p(t) \leq 0, t \in \mathbb{R}_+\). Assume that \((A_1), (A_3), (A_6)\) and \((A_7)\) hold. If any one of the following conditions

\[
(A_{10}) \int_{T}^{\infty} q(t)H(F^+(t-\sigma))dt = \infty, \quad T > 0
\]

\[
(A_{11}) \int_{T}^{\infty} q(t)H(F^-(t+\tau-\sigma))dt = \infty, \quad T > 0
\]

\[
(A_{12}) \int_{T}^{\infty} q(t)H(F^-(t-\sigma))dt = \infty, \quad T > 0
\]

\[
(A_{13}) \int_{T}^{\infty} q(t)H(F^+(t+\tau-\sigma))dt = \infty, \quad T > 0
\]

holds, then conclusion the Theorem 2.2 is true.

**Proof.** On the contrary, we proceed as in the proof of the Theorem 2.2 to conclude that \(W(t)\) and \(r(t)W'(t)\) are of constant sign on \([t_3, \infty)\). Assume that \(w'(t) < 0\) for \(t \geq t_3\). Then as in Theorem 2.2, we find that \(w(t) < 0\) and \(\lim_{t \to \infty} w(t) = -\infty\). So, there exists \(t_3 > t_2\) such that \(z(t) < F(t)\) for \(t \geq t_3\).

If \(z(t) > 0\), then \(F(t) > 0\) which is not possible. Hence, \(z(t) < 0\) and \(z(t) < F(t)\) for \(t \geq t_3\). On the other hand, \(z(t) < 0\) for \(t \geq t_3\) implies that

\[ x(t) < -p(t)x(t-\tau) \leq x(t-\tau) \leq x(t-2\tau) \leq \ldots \leq x(t_3), \]

that is, \(x(t)\) is bounded on \([t_3, \infty)\). Consequently, \(\lim_{t \to \infty} w(t)\) exists, and we get a contradiction. Therefore, \(w'(t) > 0\) for \(t \geq t_3\). Here we consider the cases: \(w(t) < 0, r(t)w'(t) > 0\) and \(w(t) > 0, r(t)w'(t) > 0\) on \([t_3, \infty), t_3 > t_2\). With the former case \(w(t) < 0\), we get \(z(t) < F(t)\) of course \(\lim_{t \to \infty} r(t)w'(t)\) exists. If \(z(t) > 0\) then \(F(t) > 0\), and we get a contradiction. Hence, \(z(t) < 0\). Clearly, \(-z(t) > -F(t)\) implies that

\[ -z(t) > \max\{0,-F(t)\} = F^-(t). \]

Therefore, for \(t \geq t_3\)
\[-x(t - \tau) \leq p(t)x(t - \tau) \leq z(t) \leq -F^-(t),\]

that is, \(x(t - \sigma) > F^-(t + \tau - \sigma), \ t \geq t_4 > t_3\) and (2.3) reduce to

\[(r(t)w'(t))' + q(t)H(F^-(t + \tau - \sigma)) \leq 0\]

for \( t \geq t_4\). Integrating the last inequality from \( t_4 \) to \( +\infty\), we obtain

\[\int_{t_4}^{\infty} q(t)H(F^-(t + \tau - \sigma)) \, dt < \infty\]

which contradicts \((A_1)\). With the later case, it follows that \(z(t) > F(t)\). If \(z(t) < 0\), then \(F(t) < 0\) which is absurd. Therefore, \(z(t) > 0\) and \(z(t) \leq x(t)\) for \( t \geq t_3 > t_2\). In this case, \(\lim_{t \to \infty} r(t)w'(t)\) exists. Since, \(F^+(t) = \max\{F(t), 0\} < z(t) \leq x(t)\) for \( t \geq t_3\), then (2.3) can be viewed as

\[(r(t)w'(t))' + q(t)H(F^+(t - \sigma)) \leq 0,\]

Integrating the above inequality from \( t_3 \) to \( +\infty\), we get

\[\int_{t_3}^{\infty} q(t)H(F^+(t - \sigma)) \, dt < \infty\]

which is a contradiction to \((A_{10})\). The case \(x(t) < 0\) for \( t \geq t_0\) is similar. Hence, the proof of the theorem is complete.

**Theorem 2.4** Let \(-\infty < -p \leq p(t) \leq -1, t \in \mathbb{R}_+, p > 0\). Assume that \((A_1), (A_3), (A_6), (A_7), (A_{10})\) and \((A_{12})\) hold. Furthermore, assume that

\[(A_4) \int_{t}^{\infty} q(t)H\left(\frac{1}{p}F^-(t + \tau - \sigma)\right) \, dt = \infty, \ T > 0\]

and

\[(A_{15}) \int_{t}^{\infty} q(t)H\left(\frac{1}{p}F^-(t + \tau - \sigma)\right) \, dt = \infty, \ T > 0.\]

Then every bounded solution of (1.1) oscillates.

**Proof.** The proof of the theorem can be followed from the proof of the Theorem 2.3. Hence the details are omitted.

**Theorem 2.5** Let \(0 \leq p(t) \leq p < \infty, t \in \mathbb{R}_+\). Assume that \((A_1), (A_3) - (A_7)\) hold. If

\[(A_6) \int_{t_4}^{\infty} Q(t) \, dt = \infty\]

hold, where \(Q(t)\) is defined in Theorem 2.2, then condition of the Theorem 2.2 is true.

**Proof.** On the contrary, we proceed as in the proof of the Theorem 2.2 to conclude that \(w(t)\) and \(r(t)w'(t)\) are of constant sign on \([t_2, \infty)\). Let’s consider the case \(w(t) > 0\), \(r(t)w'(t) > 0\) for \( t \geq t_2\). Since \(w(t)\) is nondecreasing, then there exists a constant \(\alpha > 0\) such that \(w(t) \geq \alpha\), implies that \(z(t) \geq \alpha + F(t)\), that is, \(z(t) - \alpha > F(t)\). If \(z(t) - \alpha < 0\), then \(F(t) < 0\), so we obtain a contradiction. Hence \(z(t) - \alpha > 0\). Consequently, \(z(t) - \alpha \geq \max\{F(t), 0\} = F^+(t)\), implies that

\[z(t) \geq F^+(t) + \alpha > \alpha.\]

From (2.4), it follows that

\[(r(t)w'(t))' + \lambda Q(t)H(\alpha) < 0.\]
for \( t \geq t_3 > t_2 + \sigma \). In this case, \( \lim_{{t \to \infty}} r(t)w'(t) \) exists. Integrating the last inequality from \( t_3 \) to \( +\infty \), we get

\[
\lambda H(\alpha) \left[ \int_3^T Q(t) dt \right] < \infty,
\]

a contradiction to \((A_4)\). Rest of the proof follows from the proof of the Theorem 2.2. Hence the proof of the theorem is complete.

**Theorem 2.6** Let \(-1 \leq p(t) \leq 0, t \in \mathbb{R}_+\). Assume that \((A_1),(A_2),(A_6),(A_7),(A_{14})\) and \((A_{17})\) hold. Furthermore assume that

\[
(A_{17}) \int_0^\infty q(t) dt = \infty \quad \text{holds, then conclusion of the Theorem 2.2 is true.}
\]

**Proof.** On the contrary, we proceed as in the proof of the Theorem 2.3 to conclude that \( w(t) \) and \( r(t)w'(t) \) are of constant sign on \([t_2, \infty)\). Let’s consider the case \( w(t) > 0, r(t)w'(t) > 0 \) for \( t \geq t_2 \). Proceeding as in the proof of the Theorem (2.5) we have obtained \( z(t) \geq F^+(t) + \alpha > \alpha \). Since \( z(t) \leq x(t) \), implies that \( x(t) \geq z(t) > \alpha \). From (2.3), it follows that

\[
(r(t)w'(t))' + q(t)H(\alpha) < 0
\]

for \( t \geq t_3 > t_2 + \sigma \). Integrating the last inequality from \( t_3 \) to \( t \), we have a contradiction to \((A_{17})\). Rest of the proof follows from the proof of the Theorem 2.3. Hence the theorem is proved.

**Theorem 2.7** Let \(-\infty < -p \leq p(t) \leq -1, t \in \mathbb{R}_+\). If the conditions \((A_1),(A_2),(A_6),(A_7),(A_{14}),(A_{15})\) and \((A_{17})\) are satisfied, then conclusion of the Theorem 2.4 is true.

**Proof.** The proof of the theorem can be followed from the proof of the Theorem 2.4 and Theorem 2.6. Hence the details are omitted.

In the following, we establish sufficient conditions for oscillation of all solution of (1.1) under the assumption \((A_2)\).

**Remark 2.8** If we denote \( R(t) = \int_t^\infty \frac{ds}{r(s)} \), then \( \int_0^\infty \frac{dt}{r(t)} < \infty \) implies that \( R(t) \to 0 \) as \( t \to \infty \), since \( R(t) \) is nonincreasing.

**Theorem 2.9** Let \( 0 \leq p(t) \leq p < \infty, t \in \mathbb{R}_+\). Assume that \((A_2) - (A_5)\) hold. If

\[
(A_{18}) \int_T^\infty \frac{1}{r(t)} \left[ \int_1^T Q(s)H(F^+(s) - \sigma)) ds \right] dt = \infty, T, T_1 > 0,
\]

then conclusion of the Theorem 2.2 is true, where \( Q(t) \) is defined in Theorem 2.2.

**Proof.** Let \( x(t) \) be a non-oscillatory solution of (1.1). Proceeding as in Theorem 2.2, we get (2.3) for \( t \geq t_1 \). In what follows, \( r(t)w'(t) \) and \( w(t) \) are monotonous functions on \([t_2, \infty), t_2 > t_1\).

Consider the case when \( r(t)w'(t) < 0, w(t) > 0 \) for \( t \geq t_2 \). Then, for \( s \geq t > t_2 \), \( r(s)w'(s) \leq r(t)w'(t) \) implies that \( w'(s) \leq \frac{r(t)w'(t)}{r(s)} \), that is,

\[
w(s) \leq w(t) + r(t)w'(t) \int_s^t \frac{d\theta}{r(\theta)}
\]
Since, \( r(t)w'(t) \) is nonincreasing, then there exists a constant \( C > 0 \) such that \( r(t)w'(t) \leq -C \) for \( t \geq t_2 \). As a result, \( w(s) \leq w(t) - C \int_{s}^{t} \frac{d\theta}{r(\theta)} \). As \( s \to \infty \), it follows that \( 0 \leq w(t) - CR(t) \) for \( t \geq t_2 \). Therefore, \( z(t) \geq F(t) + CR(t) \) implies that \( z(t) - CR(t) \geq F(t) \). If \( z(t) - CR(t) < 0 \), then \( F(t) < 0 \), which is a contradiction. Hence \( z(t) - CR(t) > 0 \) and hence \( z(t) - CR(t) \geq F^+(t) \), that is, \( z(t) \geq CR(t) + F^+(t) \). Consequently, (2.4) reduce to

\[
(r(t)w'(t))' + H(p)(r(t-\tau)w'(t-\tau))' + \lambda Q(t)H(F^+(t-\sigma)) \leq 0
\]

for \( t \geq t_3 > t_2 \). Integrating the above inequality from \( t_3 \) to \( t (> t_3) \), we obtain

\[
[r(s)w'(s)]_{t_3}^{t} + H(p)[r(s-\tau)w'(s-\tau)]_{t_3}^{t} + \lambda \left[ \int_{t_3}^{t} Q(s)H(F^+(s-\sigma))ds \right] \leq 0,
\]

that is,

\[
\lambda \int_{t_3}^{t} Q(s)H(F^+(s-\sigma))ds \leq -[r(s)w'(s) + H(p)[r(s-\tau)w'(s-\tau)]_{t_3}^{t}]
\]

\[
\leq -(1 + H(p))r(t)w'(t)
\]

implies that

\[
\frac{\lambda}{1 + H(p)} \int_{t_3}^{t} \frac{1}{r(t)} \left[ \int_{t_3}^{t} Q(s)H(F^+(s-\sigma))ds \right] dt \leq -[w(t)]_{t_3}^{t}.
\]

Further integration of the above inequality, we obtain that

\[
\frac{\lambda}{1 + H(p)} \int_{t_3}^{t} \frac{1}{r(t)} \left[ \int_{t_3}^{t} Q(s)G(F^+(s-\sigma))ds \right] dt \leq -[w(t)]_{t_3}^{t}.
\]

Since \( w(t) \) is bounded and monotonic, then it follows that

\[
\int_{t_3}^{t} \frac{1}{r(t)} \left[ \int_{t_3}^{t} Q(s)H(F^+(s-\sigma))ds \right] dt < \infty,
\]

a contradiction to \((A_{43})\). The rest of the proof follows from the proof Theorem 2.2. Hence the proof of the theorem is complete.

**Theorem 2.10** Let \(-1 \leq p(t) \leq 0, \ t \in \mathbb{R}_1 \). Assume that \((A_2), (A_3), (A_6), (A_7)\) and \((A_{10}) - (A_{13})\) hold. Furthermore, assume that

\[
(A_{19}) \int_{t}^{\infty} \frac{1}{r(t)} \int_{t}^{t_1} q(s)H(F^+(s+\tau-\sigma))dsdt = \infty,
\]

\[
(A_{20}) \int_{t}^{\infty} \frac{1}{r(t)} \int_{t}^{t_1} q(s)H(F^-(s+\tau-\sigma))dsdt = \infty,
\]

\[
(A_{21}) \int_{t}^{\infty} \frac{1}{r(t)} \int_{t}^{t_1} q(s)H(F^+(s-\sigma))dsdt = \infty
\]

and
\[(A_{22}) \int_{r(T)}^{\infty} \frac{1}{r(t)} \int_{r(t)}^{\infty} q(s)H(F^-(s-\sigma))dsdt = \infty, \]

where \(T, T_1 > 0\). Then conclusion of the Theorem 2.2 is true.

**Proof.** For contrary, let \(x(t)\) be a nonoscillatory solution of (1.1). Then proceeding as in Theorem 2.3 we obtain that \(w(t)\) and \(r(t)w'(t)\) are of one sign on \([t_3, \infty)\). If \(w(t) < 0\) and \(r(t)w'(t) < 0\) for \(t \geq t_3 > t_2\), then we use the same type of argument as in Theorem 2.3 to get that \(x(t)\) is bounded, that is, \(\lim_{t \to \infty} w(t)\) exists. Clearly, \(z(t) < 0\), implies that \(-z(t) > -F(t)\) and thus \(-z(t) > F^-(t)\). Therefore, for \(t \geq t_3\)

\[-x(t-\tau) \leq p(t)x(t-\tau) \leq z(t) < -F^-(t).\]

Consequently, \(x(t-\sigma) > F^-(t+\tau-\sigma)\), \(t \geq t_4 > t_3\), and (2.3) yield

\[(r(t)w'(t))' + q(t)H(F^-(t+\tau-\sigma)) \leq 0\]

for \(t \geq t_4\). Integrating the preceding inequality from \(t_4\) to \(t( > t_4)\), we obtain

\[\int_{t_4}^{t} q(t)H(F^-(t+\tau-\sigma)) < -r(t)w'(t),\]

that is,

\[\frac{1}{r(t)} \left[ \int_{t_4}^{t} q(t)H(F^-(t+\tau-\sigma)) \right] < -w'(t).\]

Further integration of the last inequality we find

\[\int_{t_4}^{\infty} \frac{1}{r(t)} \left[ \int_{t_4}^{t} q(t)H(F^-(t+\tau-\sigma)) \right] < \infty\]

which contradicts \((A_{20})\). If \(w(t) > 0\) and \(r(t)w'(t) < 0\) for \(t \geq t_3\), then following to Theorem 2.9 we find

\[z(t) \geq F^+(t) + CR(t) \geq F^{+}(t)\]

and \(z(t) > 0\), that is, \(x(t) \geq F^{+}(t)\). The remaining case follows from previous one and the rest of the proof can similarly be dealt with the proof of Theorem 2.3. Hence, the theorem is proved.

**Theorem 2.11** Let \(-\infty < -p \leq p(t) \leq -1, t \in \mathbb{R}_+\). Assume that \((A_2), (A_3), (A_5), (A_9) - (A_{13}), (A_{21})\) and \((A_{22})\) hold. Furthermore, assume that

\[(A_{23}) \int_{t}^{\infty} \frac{1}{r(t)} \int_{r(t)}^{\infty} q(s)H(\frac{1}{p} F^+(s+\tau-\sigma))dsdt = \infty\]

and

\[(A_{24}) \int_{t}^{\infty} \frac{1}{r(t)} \int_{r(t)}^{\infty} q(s)H(\frac{1}{p} F^-(s+\tau-\sigma))dsdt = \infty,\]

where \(T, T_1 > 0\). Then conclusion of the Theorem 2.4 is true.

**Proof.** The proof of the theorem can be followed from the proof of the Theorem 2.10. Hence the details are omitted.

**Theorem 2.12** Let \(0 \leq p(t) \leq p < \infty, \ t \in \mathbb{R}_+\). Assume that \((A_2) - (A_7), (A_{10})\) and \((A_{18})\) holds, then conclusion of the Theorem 2.2 is also true.

**Proof.** Proof of the theorem follows from the proof of the Theorems 2.5 and Theorem 2.9 and hence the details are omitted.
Theorem 2.13 Let \(-1 \leq p(t) \leq 0, \ t \in \mathbb{R}_+\). Assume that \((A_2), (A_4), (A_6), (A_7), (A_{11}), (A_{13}), (A_{17})\) and \((A_{10}) - (A_{22})\) hold, then also conclusion of the Theorem 2.2 is true.

Proof. The proof of the theorem can be followed from the proof of the Theorem 2.6 and Theorem 2.10. Hence the details are omitted.

Theorem 2.14 Let \(-\infty < -p \leq p(t) \leq -1, \ t \in \mathbb{R}_+\). Assume that \((A_2), (A_4), (A_6), (A_7), (A_{11}), (A_{13}), (A_{17})\) and \((A_{21}) - (A_{24})\) hold. Then conclusion of the Theorem 2.4 is true.

Proof. The proof of the theorem can be followed from the proof of the Theorem 2.7 and Theorem 2.11. Hence the details are omitted.

3. Oscillation properties of Eq. (1.2)

In this section, sufficient conditions are obtained for oscillation and asymptotic behaviour of solutions for nonlinear second order neutral differential equations of the form (1.2). We need the following conditions for this work in the sequel.

\[
(A_{25})\int_{\tau}^{\infty} H[C(1 - p(t - \sigma))]q(t)\,dt = \infty, \ T, C > 0; \\
(A_{26})\int_{t}^{\infty} \frac{1}{r(t)} \int_{\tau}^{\infty} Q(s)H(CR(s - \sigma))\,dsdt = \infty, \ T, T_1, C > 0; \\
(A_{27})\int_{t}^{\infty} \frac{1}{r(t)} \int_{t}^{\infty} q(s)H(CR(s - \sigma))\,dsdt = \infty, \ T, T_1, C > 0; \\
(A_{28})\int_{0}^{\infty} \frac{1}{r(t)} \int_{0}^{t} q(s)\,dsdt = \infty.
\]

Theorem 3.1 Let \(0 \leq p(t) \leq p < \infty, \ t \in \mathbb{R}_+\). Assume that \((A_4), (A_4) - (A_6)\) and \((A_{10})\) hold. Then every solution of the equation (1.2) is oscillatory.

Proof. Suppose for contrary that \(x(t)\) is a nonoscillatory solution of equation (1.2). Then there exists \(t_0 \geq \rho\) such that \(x(t) > 0\) or \(< 0\) for \(t \geq t_0\). Assume that \(x(t) > 0\) and \(x(t - \sigma) > 0\) for \(t \geq t_0\). Setting \(z(t) = x(t) + p(t)x(t - \tau), \ t \geq 0\)

From (1.2), it follows that

\[
(r(t)z'(t))' = -q(t)H(x(t - \sigma)) < 0,
\]

hold for \(t \geq t_1 > t_0\). Consequently, \(r(t)z'(t)\) is nonincreasing and \(z'(t), \ z(t)\) are of constant sign on \([t_2, \infty), t_2 > t_1\). Let \(r(t)z'(t) < 0\) for \(t \geq t_2\). Then we can find \(K > 0\) and a \(t_3 > t_2\) such that \(r(t)z'(t) \leq -K\) for \(t \geq t_3\). Integrating the relation \(z'(t) \leq -\frac{K}{r(t)}, \ t \geq t_3\) from \(t_3\) to \(t > t_3\) and obtain

\[
z(t) \leq z(t_3) - K\left[\int_{t_3}^{t} \frac{ds}{r(s)}\right]
\]

\[
\rightarrow -\infty, \ \text{as} \ \ t \rightarrow \infty,
\]
due to \((A_4)\), a contradiction to the fact that \(z(t) > 0\) for \(t \geq t_1\). Hence, \(r(t)z'(t) > 0\) for \(t \geq t_2\). As a result, \(z(t)\) is nondecreasing on \([t_2, \infty)\). So, there exists \(C > 0\) and a \(t_3 > t_2\) such that \(z(t) \geq C\) for \(t \geq t_3\). We note that \(\lim_{t \rightarrow \infty} r(t)z'(t)\) exists. Using (1.2), it follows that

\[
(r(t)z'(t))' + q(t)H(x(t - \sigma)) + H(p)[(r(t - \tau)z'(t - \tau))'] + q(t - \tau)H(x(t - \tau - \sigma))] = 0.
\]
Upon using (A_4) and (A_5), the last equation becomes
\[ 0 \geq (r(t)z'(t))' + H(p)(r(t-\tau)z'(t-\tau))' + \lambda Q(t)H(z(t-\sigma)), \]
where \( z(t) \leq x(t) + px(t-\tau) \). Consequently, there exists \( t_4 > t_3 \) such that
\[ (r(t)z'(t))' + H(p)(r(t-\tau)z'(t-\tau))' + \lambda H(C)Q(t) \leq 0 \]
for \( t \geq t_4 \). Integrating (3.4) from \( t_4 \) to \( t > t_4 \), then
\[ \lambda H(C) \left[ \int_{t_4}^{t} Q(s)ds \right] \leq -[r(s)z'(s)]_{t_4}^{t} - H(p)[r(s-\tau)z'(s-\tau)]_{t_4}^{t} < \infty, \quad \text{as} \; t \to \infty, \]
which a contradiction due to the assumption (A_6).

If \( x(t) < 0 \) for \( t \geq t_0 \), then we set \( y(t) = -x(t) \) for \( t \geq t_0 \) in (1.2) and using (A_6) we find
\[ (r(t)(y(t) + p(t)y(t-\tau))')' + q(t)H(y(t-\sigma)) = 0, \]
then proceeding as above, we find a same contradiction. This completes the proof of the theorem.

**Remark 3.2** Indeed, we don’t need (A_4), if we restrict \( 0 \leq p(t) < 1, t \in \mathbb{R}_+ \). When \( z(t) \) is nondecreasing, it happens that
\[ z(t) - p(t)z(t-\tau) = x(t) + p(t)x(t-\tau) - p(t)x(t-\tau) - p(t)p(t-\tau)x(t-2\tau) = x(t) - p(t)p(t-\tau)x(t-2\tau) < x(t), \]
that is,
\[ (1 - p(t))z(t) < x(t). \]

Therefore, (1.2) can be written as
\[ (r(t)z'(t))' + q(t)H[(1 - p(t-\sigma))(z(t-\sigma))] \leq 0. \]
Hence or otherwise, we have proved the following theorem:

**Theorem 3.3** Let \( 0 \leq p(t) < 1, t \in \mathbb{R}_+ \). Assume that (A_4), (A_5) and (A_25) holds, then conclusion of the Theorem 3.1 is true.

**Theorem 3.4** Let \( -1 \leq p(t) \leq 0, t \in \mathbb{R}_+ \). If (A_4), (A_5) and (A_17) hold, then every unbounded solution of (1.2) oscillates.

**Proof.** Let on the contrary that \( x(t) \) be a unbounded solution of (1.2) on \([t_0, \infty), \; t_0 > p \). Proceeding as in Theorem 3.1, it concludes that \( r(t)z'(t) \) is nonincreasing on \([t_2, \infty) \). Since \( z(t) \) is monotonic, then there exists \( t_3 > t_2 \) such that \( z(t) > 0 \) or \( < 0 \) for \( t \geq t_3 \). Indeed, \( z(t) < 0 \) for \( t \geq t_3 \) implies that \( x(t) \leq x(t-\tau) \), and hence
\[ x(t) \leq x(t-\tau) \leq x(t-2\tau) \leq ... \leq x(t_3), \]
that is, \( x(t) \) is bounded, which is absurd. Hence, \( z(t) > 0 \) for \( t \geq t_3 \). Suppose that \( r(t)z'(t) > 0 \) for \( t \geq t_3 \). Clearly, \( z(t) \leq x(t) \) implies that
\[ (r(t)z'(t))' + q(t)H(z(t-\sigma)) \leq 0 \]
for $t \geq t_3$. On the other hand, $z(t)$ is nondecreasing implies that, there exist $C > 0$ and a $t_4 > t_3$ such that $z(t) \geq C$ for $t \geq t_4$. Consequently, for $t_5 > t_4 + \sigma$, it follows from (3.5) that

$$(r(t)z'(t))' + H(C)q(t) \leq 0, t \geq t_5$$

Integrating the last inequality from $t_5$ to $t$ ($> t_5$), we have

$$H(C)\left[\int_{t_5}^{t} q(s)ds\right] \leq -[r(s)z'(s)]_{t_5}$$

$$< \infty, \text{ as } t \to \infty,$$

which is a contradiction to $(A_{1\gamma})$. Hence, $r(t)z'(t) < 0$ for $t \geq t_3$. Rest of the theorem follows from Theorem 3.1. Thus, the proof of the theorem is complete.

**Theorem 3.5** Let $-1 < -p \leq p(t) \leq 0$, $t \in \mathbb{R}_+$ and $p > 0$. If all the assumptions of Theorem 3.4 hold, then every solution of (1.2) either oscillates or converges to zero as $t \to \infty$.

**Proof.** Proceeding as in the proof of Theorem 3.1, we have obtained (3.2) and hence $r(t)z'(t)$ is nonincreasing on $[t_2, \infty)$. Therefore, $z(t)$ is monotonic on $[t_3, \infty), t_3 > t_2$. So we have four cases namely:

(i) $z(t) > 0$, $r(t)z'(t) > 0$,

(ii) $z(t) > 0$, $r(t)z'(t) < 0$,

(iii) $z(t) < 0$, $r(t)z'(t) > 0$,

(iv) $z(t) < 0$, $r(t)z'(t) < 0$.

Using the arguments as in the proof of Theorems 3.1 and 3.4, we get contradictions to $(A_i)$ and $(A_{1\gamma})$ when the Cases (ii) and Case (i) respectively. Since $z(t) < 0$ implies that $x(t)$ is bounded, that is, $z(t)$ is bounded, then the Case (iv) is not possible due to Theorem 3.1: $z'(t) < 0$, implies that $\lim_{t \to \infty} z(t) = -\infty$. Consequently, the Case (iii) holds for $t \geq t_3$. In this case, $\lim_{t \to \infty} z(t)$ exits. As a result,

$$0 \geq \lim_{t \to \infty} z(t) = \limsup_{t \to \infty} z(t)$$

$$= \limsup_{t \to \infty} (x(t) + p(t)x(t-\tau))$$

$$\geq \limsup_{t \to \infty} (x(t) - p(t)x(t-\tau))$$

$$\geq \limsup_{t \to \infty} x(t) + \liminf_{t \to \infty} (-p(t)x(t-\tau))$$

$$= (1 - p) \limsup_{t \to \infty} x(t)$$

implies that $\limsup_{t \to \infty} x(t) = 0$ [since $1 - p > 0$] and hence $\liminf_{t \to \infty} x(t) = 0$. Thus $\lim_{t \to \infty} x(t) = 0$. The case $x(t) < 0$ is similar dealt with. This completes the proof of the theorem.

**Theorem 3.6** Let $-\infty < -p_1 \leq p(t) \leq -p_2 < -1$, $p_1, p_2 > 0$ and $t \in \mathbb{R}_+$. If $(A_i), (A_6)$ and $(A_{1\gamma})$
hold, then every bounded solution of (1.2) either oscillates or converges to zero as $t \to \infty$.

**Proof.** Suppose on the contrary that $x(t)$ is a solution of (1.2) which is bounded on $[t_0, \infty)$, $t_0 > \rho$. Using the same type of reasoning as in Theorem 3.1, we have that $z(t)$ and $z(t)$ are of one sign on $[t_2, \infty)$ and have four possible cases like as in Theorem 3.5. By Theorem 3.1, Case(iv) is not possible because of $(A_5)$ and bounded $z(t)$.

Also, same is true for the Case(ii). Case(i) follows from the proof of the Theorem 3.4. Consider Case(iii). In this case, $\lim_{t \to \infty} z(t)$ exists. Let $\lim_{t \to \infty} z(t) = \beta, \beta \in (-\infty, 0]$. Assume that $-\infty < \beta < 0$. Then there exists $\alpha < 0$ and $t_3 > t_2$ such that $z(t + \tau - \sigma) < \alpha$, for $t \geq t_3$. Hence, $z(t) \geq p(t) x(t - \tau) \geq -p_1 x(t - \tau)$ implies that $x(t - \sigma) \geq -p_1^{-1} \alpha > 0$, for $t \geq t_3$. Consequently, (1.2) becomes

$$
(r(t)z'(t))' + H(-p_1^{-1} \alpha)q(t) \leq 0, \tag{3.6}
$$

for $t \geq t_3$. Integrating (3.6) from $t_3$ to $t(t > t_3), we get

$$
H(-p_1^{-1} \alpha) \left[ \int_{t_3}^t q(s)ds \right] \leq -[r(s)z'(s)]_{t_3} < \infty, \text{ as } t \to \infty,
$$

which is a contradiction to $(A_{15})$. Ultimately, $\beta = 0$. Hence,

$$
0 = \lim_{t \to \infty} z(t) = \operatorname{liminf}_{t \to \infty} z(t)
\leq \liminf_{t \to \infty} (x(t) - p_2 x(t - \tau))
\leq \limsup_{t \to \infty} x(t) + \liminf_{t \to \infty} (-p_2 x(t - \tau))
= (1 - p_2) \limsup_{t \to \infty} x(t)
$$

implies that $\limsup_{t \to \infty} x(t) = 0$ [for $1 - p_2 < 0$]. Thus, $\liminf_{t \to \infty} x(t) = 0$ and hence $\lim_{t \to \infty} x(t) = 0$. Therefore, any solution $x(t)$ of (1.2) converges to zero. The case $x(t) < 0$ is similar. This completes the proof of the theorem.

In the following, we establish sufficient conditions for oscillation of all solutions of (1.2) under the assumption that

$$(A_5) \quad \text{ and } (A_7).$$

**Theorem 3.7** Let $0 \leq p(t) \leq p < \infty, \ t \in \mathbb{R}_+. \ Assume that (A_2), (A_4) - (A_6), (A_{16})$ and $(A_{26})$ holds, then also conclusion of the Theorem 3.1 is true, where $Q(t)$ is defined in Theorem 3.1.

**Proof.** On the contrary, we proceed as in Theorem 3.1 to obtain (3.2), for $t \geq t_1$ and $r(t)z'(t)$ is non increasing on $[t_2, \infty), \ t_2 > t_1$. The case $r(t)z'(t) > 0$ is for $t \geq t_0$ is same as in Theorem 3.1 and gives a contradiction due to $(A_{15})$. Let’s suppose that $r(t)z'(t) < 0$, for $t \geq t_2$. Therefore, for $s \geq t > t_2$, $r(s)z'(s) \leq r(t)z'(t)$ implies that

$$
z'(s) \leq \frac{r(t)z'(t)}{r(s)}.
$$

Consequently,

$$
z(s) \leq z(t) + r(t)z'(t) \int_t^s \frac{d\theta}{r(\theta)}.
$$
Because of \( r(t)z'(t) \) is nonincreasing, we can find a constant \( C > 0 \) such that \( r(t)z'(t) \leq -C \) for \( t \geq t_2 \). As a result, 
\[
(3.3) 
\]
\[
0 \leq z(t) - CR(t) \quad \text{for} \quad t \geq t_2.
\]
Using the above fact in (3.3), we get
\[
(r(t)z'(t))' + H(p)(r(t - \tau)z'(t - \tau))' + \lambda Q(t)H(CR(t - \sigma)) \leq 0,
\]
for \( t \geq t_3 > t_2 \). Integrating (3.7) from \( t_3 \) to \( t > t_3 \), we obtain
\[
[r(s)z'(s)]_{t_3}^{t} + H(p)[r(s - \tau)z'(s - \tau)]_{t_3}^{t} + \lambda \int_{t_3}^{t} Q(s)H(CR(s - \sigma))ds \leq 0,
\]
that is,
\[
\lambda \int_{t_3}^{t} Q(s)H(CR(s - \sigma))ds \leq -[r(s)z'(s) + H(p)(r(s - \tau)z'(s - \tau))]_{t_3}^{t}
\]
\[
\leq -(1 + H(p))r(t)z'(t)
\]
implies that
\[
\frac{\lambda}{1 + H(p)} \frac{1}{r(t)} \left[ \int_{t_3}^{t} Q(s)H(CR(s - \sigma))ds \right] \leq -z'(t).
\]
Again integrating the last inequality, we obtain that
\[
\frac{\lambda}{1 + H(p)} \frac{1}{r(t)} \left[ \int_{t_3}^{t} Q(s)H(CR(s - \sigma))ds \right] dt \leq -[z(t)]_{t_3}^{t}.
\]
Since \( z(t) \) is bounded and monotonic, then it follows that
\[
\int_{t_3}^{t} \frac{1}{r(t)} \left[ \int_{t_3}^{t} Q(s)H(CR(s - \sigma))ds \right] dt < \infty,
\]
which is a contradiction to \((A_{2b})\). The case \( x(t) < 0 \) is similar dealt with. This completes the proof of the theorem.

**Theorem 3.8** Let \( -1 \leq p(t) \leq 0, \quad t \in \mathbb{R}_+ \). Assume that \((A_2),(A_3),(A_{17})\) and \((A_{27})\) hold, then conclusion of the Theorem 3.4 is true.

**Proof.** Proof of the theorem follows from the proof of the Theorems 3.4 and 3.7 and hence the details are omitted.

**Theorem 3.9** Let \(-1 < -p \leq p(t) \leq 0, \quad t \in \mathbb{R}_+ \) and \( p > 0 \). If all the conditions of Theorem 3.8 are satisfied, then conclusion of the Theorem 3.5 is true.

**Proof.** The proof of the theorem follows from the proof of Theorems 3.5 and 3.8. Hence, the proof of the theorem is complete.

**Theorem 3.10** Let \(-\infty < -p_1 \leq p(t) \leq -p_2 < -1, \quad t \in \mathbb{R}_+ \) and \( p_1, p_2 > 0 \). Assume that \((A_2),(A_3),(A_{17}),(A_{27})\) and \((A_{28})\) hold, then conclusion of the Theorem 3.6 is true.

**Proof.** Proceeding as in the proof of the Theorem 3.6 we have four possible cases for \( t \geq t_2 \). First three cases are similar to the proof of Theorem 3.6. Hence, we consider the Case \((iv)\) only. Using the same type of reasoning as in the Case \((iii)\), we get (3.6) and hence
\[
H(-p_1^{-1} \alpha \left[ \int_{t_3}^{t} q(s)ds \right]) \leq -r(t)z'(t).
\]
Therefore,
\[
H(-p_{-1}z(t)) - \int_0^t \frac{1}{r(s)} \left[ \int_s^\infty q(\theta) d\theta \right] ds dt \leq -[z(t)]_1^\infty \leq -z(u) < \infty, \text{ as } u \to \infty,
\]
due to bounded and monotonic \( z(t) \), a contradiction. The case \( x(t) < 0 \) is similar. Rest of the theorem follows from the proof of Theorem 3.6. This completes the proof of the theorem.

**Remark 3.11** In Theorem 2.2 - Theorem 3.10, \( H \) could be linear, sublinear or superlinear.

**4. Existence of positive solution**

In this section, necessary conditions are obtained to show that equation (1.1) admits a positive bounded solution for various ranges of \( p(t) \).

**Theorem 4.1** Let \( p \in C([T, +\infty)) \) and assume that \((A_4)\) hold. If
\[
(A_{29}) \int_0^\infty \frac{1}{r(s)} \left[ \int_s^\infty q(\theta) d\theta \right] ds < \infty,
\]
then (1.1) admits a positive bounded solution.

**Proof.** Let \((i)\) \(-1 < -p \leq p(t) \leq 0, \ t \in \mathbb{R}_+ \) and \( p > 0 \). Due to \((A_{29})\), it is possible to find a \( T > \rho \) such that
\[
\int_T^\infty \frac{1}{r(s)} \left[ \int_s^\infty q(\theta) d\theta \right] ds < \frac{1-p}{10H(1)}.
\]
We consider the set
\[
M = \{ x : x \in C([T - \rho, +\infty), \mathbb{R}), x(t) = 0 \text{ for } t \in [T - \rho, T] \text{ and } \frac{1-p}{20} \leq x(t) \leq 1 \}
\]
and define \( \Phi : M \to C([T - \rho, +\infty), \mathbb{R}) \) by the formula
\[
\Phi x(t) = \begin{cases} 
0, & t \in [T - \rho, T) \\
-p(t)x(t - \tau) + \int_T^\infty \frac{1}{r(s)} \left[ \int_s^\infty q(\theta) H(x(\theta - \sigma)) d\theta \right] d\sigma + F(t) + \frac{1-p}{10}, & t \geq T,
\end{cases}
\]
where \( F(t) \) be such that \( |F(t)| \leq \frac{1-p}{20} \). For every \( x \in M \),
\[
\Phi x(t) \leq -p(t)x(t - \tau) + \int_T^\infty \frac{1}{r(s)} \left[ \int_s^\infty q(\theta) H(x(\theta - \sigma)) d\theta \right] d\sigma + F(t) + \frac{1-p}{10} + \frac{1-p}{10} - \frac{1-p}{20} - \frac{1-p}{20} - \frac{1-p}{4} < 1,
\]
and
\[
\Phi x(t) \geq F(t) + \frac{1-p}{10}.
\]
implies that $\Phi x(t) \in M$. Define $u_n : [T - \rho, +\infty) \to \mathbb{R}$ by the recursive formula
\[ u_n(t) = (\Phi u_{n-1})(t), \quad n \geq 1 \]
with the initial condition
\[ u_0(t) = \begin{cases} 0, & t \in [T - \rho, T), \quad t > T. \end{cases} \]
Inductively it is easy to verify that
\[ \frac{1 - p}{20} \leq u_{n-1}(t) \leq u_n(t) \leq 1 \]
for $t \geq T$. Therefore for $t \geq T - \rho$, $\lim_{n \to \infty} u_n(t)$ exists. Let $\lim_{n \to \infty} u_n(t) = u(t)$ for $t \geq T - \rho$. By the Lebesgue’s dominated convergence theorem, $u \in M$ and $\Phi u(t) = u(t)$, where $u(t)$ is a solution of (1.1) on $[T - \rho, \infty)$ such that $u(t) > 0$. Hence, $(A_{29})$ is necessary.

(ii) If $p(t) = -1, t \in \mathbb{R}_+$, we choose $-1 < p_2 < 0$ such that $p_2 \neq -\frac{1}{2}$. In this case, we can apply the above method. Here, we note that
\[ \int_0^t \frac{1}{r(s)} \left[ \int_s^\infty q(\theta)d\theta \right] ds < \frac{1 + 2p_2}{10H(-p_2)}, \]
\[ -\frac{1 + 2p_2}{40} \leq F(t) \leq \frac{1 + 2p_2}{20} \] and the set
\[ M = \{ x : x \in C([T - \rho, +\infty), \mathbb{R}), x(t) = 0 \text{ for } t \in [T - \rho, T] \} \]
Also, we define $\Phi : M \to C([T - \rho, +\infty), \mathbb{R})$ by
\[ \Phi x(t) = \begin{cases} 0, & t \in [T - \rho, T), \\
 x(t - \tau) + \int_{T - \rho}^t \frac{1}{r(s)} \left[ \int_s^\infty q(\theta)H(x(\theta - \sigma))d\theta \right] ds + F(t) + \frac{2 + p_2}{10}, & t \geq T. \end{cases} \]
This completes the proof of the theorem.

**Theorem 4.2** Let $p \in C(\mathbb{R}_+, [0,1])$. Let $H$ be Lipschitzian on the interval of the form $[a, b], 0 < a < b < \infty$. If $(A_{29})$ and $(A_{29})$ hold, then (1.1) admits a positive bounded solution.

**Proof.** Let $0 \leq p(t) \leq p_3 < 1$. It is possible to find $t_1 > 0$ such that
\[ \int_{t_1}^{t_1} \frac{1}{r(s)} \left[ \int_s^\infty q(\theta)d\theta \right] ds < \frac{1 - p_3}{5L}, \]
where $L = \max\{L_1, H(1)\}$, $L_1$ is the Lipschitz constant on $\left[\frac{3}{5}(1-p_3), 1\right]$. Let $F(t)$ be such that $|F(t)| < \frac{1-p_3}{10}$ for $t \geq t_2$. For $t_3 > \max\{t_1, t_2\}$, we set $Y = BC([T, \infty), \mathbb{R})$, the space of real valued continuous functions on $[t_3, \infty]$. Clearly, $Y$ is a Banach space with respect to sup norm defined by

$$
\|y\| = \sup\{|y(t)|; t \geq t_3\}.
$$

Let's define

$$
S = \{u \in Y: \frac{3}{5}(1-p_3) \leq u(t) \leq 1, t \geq t_3\}.
$$

We notice that $S$ is a closed and convex subspace of $X$. Let $\Phi: S \to S$ be such that

$$
\Phi x(t) = \begin{cases} 
\Phi x(t + \rho), & t \in [t_3, t_3 + \rho] \\
- p(t)x(t - \tau) + \frac{9 + p_3}{10} + F(t) + \int_{t}^{\infty} \frac{1}{r(s)} \left[ \int_{s}^{\infty} q(\theta) H(x(\theta - \sigma)) d\theta \right] ds, & t \geq t_3 + \rho.
\end{cases}
$$

For every $x \in Y$, $\Phi x(t) \leq F(t) + \frac{9 + p_3}{10} \leq 1$ and

$$
\Phi x(t) \geq -p(t)x(t - \tau) - H(1) \int_{t}^{\infty} \frac{1}{r(s)} \left[ \int_{s}^{\infty} q(\theta) d\theta \right] ds + F(t) + \frac{9 + p_3}{10}
$$

$$
\geq -p_3 - \frac{1-p_3}{5} - \frac{1-p_3}{10} + \frac{9 + p_3}{10} = \frac{3}{5}(1-p_3)
$$

implies that $\Phi x \in S$. Now for $x_1$ and $x_2 \in S$, we have

$$
|\Phi x_1(t) - \Phi x_2(t)| \leq p_3 |x_1(t - \tau) - x_2(t - \tau)|
$$

$$
+ \int_{t}^{\infty} \frac{1}{r(s)} \left[ \int_{s}^{\infty} q(\theta) |H(x_1(\theta - \sigma)) - H(x_2(\theta - \sigma))| d\theta \right] ds,
$$

that is,

$$
|\Phi x_1(t) - \Phi x_2(t)| \leq p_3 |x_1 - x_2|_2 + |x_1 - x_2|_4 \int_{t}^{\infty} \frac{1}{r(s)} \left[ \int_{s}^{\infty} q(\theta) d\theta \right] ds
$$

$$
\leq \left( p_3 + \frac{1-p_3}{5} \right) |x_1 - x_2|
$$

$$
= \frac{4p_3 + 1}{5} |x_1 - x_2|.
$$

Therefore, $|\Phi x_1 - \Phi x_2| \leq \frac{4p_3 + 1}{5} |x_1 - x_2|$ implies that $\Phi$ is a contraction. By using Banach's fixed point theorem, it follows that $\Phi$ has a unique fixed point $x(t)$ in $\left[\frac{3}{5}(1-p_3), 1\right]$. Hence, $\Phi x = x$ and the proof of the theorem is complete.
Remark 4.3 We can not apply Lebesgue’s dominated convergence theorem for other ranges of \( p(t) \), except \(-1 \leq p(t) \leq 0\) due to the technical difficulties arising in the method. However, we can apply Banach’s fixed point theorem to other ranges of \( p(t) \) similar to Theorem 4.2.

5. Discussion and Examples

It is worth observation that both unforced and forced equations (1.1) and (1.2) are studied keeping in view of assumptions \((A_1) - (A_{20})\). The results concerning equations (1.1) and (1.2) are oscillatory due to the analysis incorporated here. Of course the forcing term can be considered to (1.1). In [10], Santra has studied \((E_1)\) and \((E_2)\) under the assumption \( H(uv) = H(u)H(v) \) for \( u, v \in \mathbb{R}_+ \). Also proved that \( H(uv) = H(u)H(v) \), \( u, v \in \mathbb{R}_+ \) implies \( H(\bar{u}) = -H(u) \) for \( u \in \mathbb{R}_+ \). In this work, we considered a generalized condition \((A_5)\). If we take \( H(uv) = H(u)H(v) \) to this work, then some of the oscillatory conditions will be deleted \((A_{11}) \equiv (A_{14}), (A_{13}) \equiv (A_{15}), (A_{19}) \equiv (A_{23}), (A_{20}) \equiv (A_{24})\).

We conclude this section with the following examples:

Example 5.1 Consider

\[
(x(t) + x(t - \pi))^n + x(t - 2\pi) = \sin(t).
\]  

(5.1)

Here \( Q(t) = 1 \), \( \tau = \pi \), \( \sigma = 2\pi \) and \( f(t) = \sin(t) \). If we set \( F(t) = -\sin(t) \), then \( (r(t)F'(t))' = F''(t) = \sin(t) = f(t) \),

\[
F^-(t) = \begin{cases} \sin(t), & 2n\pi \leq t \leq 2n\pi + \pi \\ 0, & \text{otherwise}, \end{cases}
\]

and

\[
F^+(t) = \begin{cases} -\sin(t), & 2n\pi + \pi \leq t \leq 2n\pi + 2\pi \\ 0, & \text{otherwise}. \end{cases}
\]

Therefore

\[
F^-(t-2\pi) = \begin{cases} \sin(t), & 2n\pi + 2\pi \leq t \leq 2n\pi + 3\pi \\ 0, & \text{otherwise}, \end{cases}
\]

and

\[
F^+(t-2\pi) = \begin{cases} -\sin(t), & 2n\pi + 3\pi \leq t \leq 2n\pi + 4\pi \\ 0, & \text{otherwise}. \end{cases}
\]

For \( n = 0, 1, 2, \ldots \), we get

\[
\int_{2n\pi}^{2n\pi+4\pi} Q(t)F^+(t-2\pi)dt = \sum_{n=0}^{\infty} \int_{2n\pi+3\pi}^{2n\pi+4\pi} [-\sin(t)]dt
\]

\[
= \sum_{n=0}^{\infty} [\cos(t)]_{2n\pi+3\pi}^{2n\pi+4\pi} = +\infty.
\]

Clearly, \((A_j)\) and \((A_j) - (A_9)\) are satisfied. Hence, by Theorem 2.2, every solution of (5.1) is oscillatory. Thus, in particular, \( x(t) = \sin(t) \) is an oscillatory solution of the equation (5.1).
**Example 5.2** Consider

\[(x(t) + 2x(t - 2\pi))'' + 3\pi(t - 4\pi) = 0, \quad (5.2)\]

where \(Q(t) = 3, \quad \tau = 2\pi, \quad \sigma = 4\pi\) and \(H(x) = x\). Then every conditions of Theorem 3.1 are hold true. Hence by Theorem 3.1, every solutions of (5.2) is oscillatory. Thus in particular, \(x(t) = \cos t\) is an oscillatory solution of the equation (5.2).

**References**


